Generalized Sum and Difference Sets and d-dimensional Modular Hyperbolas

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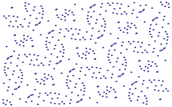

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- Twin prime conjecture: $P - P$ contains 2 infinitely often.

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- Martin and O'Bryant '07: positive percentage are sum-dominant.
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- Several ways to see new behavior usually dwarfed by large size of typical random set.
- Can choose elements equally with probability tending to 0, or can choose sets with great structure.

Goals

- Eichhorn, Khan, Stein, and Yankov [EKSY] studied modular hyperbolas:

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- Discuss tools and techniques.

Pictures

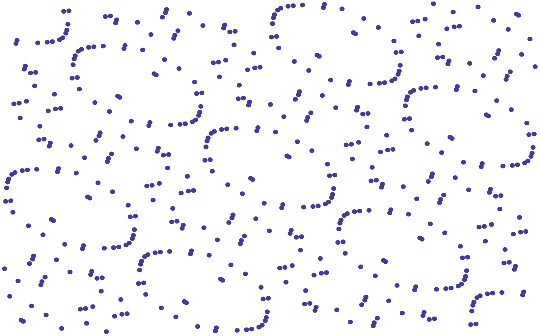

Figure : $xy \equiv 197 \bmod 2^{10}$

Pictures

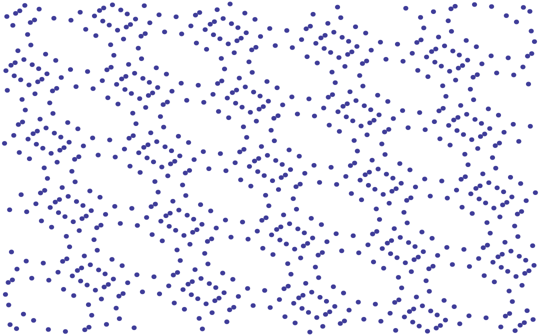

Figure : $xy \equiv 1325 \bmod 48^2$

Sums and Differences of the Coordinates of Points on Modular Hyperbolas

Dennis Eichhorn, Mizan R. Khan, Alan H. Stein, and Christian L. Yankov

Modular Hyperbolas

Definition (Modular Hyperbola)

Let a be coprime to n. A d-dimensional modular hyperbola is

$$H_d(a;n) = \{(x_1, x_2, \cdots, x_d) : x_1 \cdots x_d \equiv a \bmod n, 1 \leq x_i < n)\}.$$

[ESKY] studied $H_2(1;n)$.

Notation

We utilize the following notation:

$$\bar{D}_2(a;n) = \{x - y \bmod n : (x,y) \in H_2(a;n)\}$$

$$\bar{S}_2(a;n) = \{x + y \bmod n : (x,y) \in H_2(a;n)\}$$

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For $d > 2$ and $m \geq 1$, where m is the number of plus signs in $\pm x_1 \pm x_2 \pm \cdots \pm x_d$, let

$$\bar{S}_d(m;a;n) = \{x_1 + \cdots + x_m - \cdots - x_d \bmod n : (x_1, \cdots, x_d) \in H_d(a;n)\}.$$

[EKSY] results

Theorem (EKSY 2009)

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Theorem (EKSY 2009)

- *Found and proved explicit formulas for the cardinality of* $\bar{S}_2(1; n)$ *and* $\bar{D}_2(1; n)$.
- *Analyzed ratios of the cardinalities of* $\bar{S}_2(1; n)$ *and* $\bar{D}_2(1; n)$*, found that at least 84% of the time,* $\bar{S}_2(1; n) > \bar{D}_2(1; n)$.

$xy \equiv a \pmod{n}$
New Results

Method

Proposition 1 Generalization

Let $n = \prod_{i=1}^{m} p_i^{e_i}$ be the canonical factorization of n. Then,

$$\#\bar{S}_d(m; a; n) = \prod_{i=1}^{k} \#\bar{S}_d(m; a \bmod p_i^{e_i}; p_i^{e_i}).$$

Sketch of proof:

Consider

$$g : \bar{S}_d(m; a; n) \rightarrow \prod_{i=1}^{k} \bar{S}_d(m; a \bmod p_i^{e_i}; p_i^{e_i})$$

where

$$g(x) = (x \bmod p_1^{e_1}, \cdots, x \bmod p_k^{e_k}).$$

By Chinese remainder theorem, g is a bijection.

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5. Similar results for $\bar{S}_2(a; n)$.

Method

Theorem 3 (Stangl)

Let p be an odd prime. Then,

$$\#\{k^2 \bmod p^t\} = \frac{p^{t+1}}{2(p+1)} + (-1)^{t-1}\frac{p-1}{4(p+1)} + \frac{3}{4}.$$

In the case $p = 2$,

$$\#\{k^2 \bmod 2^t\} = \frac{2^{t-1}}{3} + \frac{(-1)^{t-1}}{6} + \frac{3}{2}, t \geq 2.$$

Method

Proposition 4

For $p \nmid c$, we have the following:

1. $p \neq 2$: If $x^2 \equiv c \mod p$ is solvable, then for every $t \geq 2$, $x^2 \equiv c \mod p^t$ has exactly 2 distinct solutions.
2. $p = 2$: If $x^2 \equiv c \mod 2^3$ is solvable, then for every $t \geq 3$, $x^2 \equiv c \mod 2^t$ has exactly 4 distinct solutions.

Method

Proposition 5

$$\bar{S}_2(a;n) = \bar{D}_2(-a;n).$$

Sketch of proof:

- If $x + y \in \bar{S}_2(a;n)$, then $(x, -y) \in H_2(-a;n)$.
- Thus $x - (-y) \in \bar{D}_2(-a;n)$.

Explicit Formulas

In the case when $p = 2$,

$$\#\bar{D}_2(a; 2^t) = \begin{cases} \frac{2^t - 4}{3} + \frac{(-1)^{t-1}}{3} + 3 & t \geq 5, a \equiv 7 \bmod 8 \\ 2^{t-3} & t \geq 5, a \equiv 1, 5 \bmod 8 \\ 2^{t-4} & t \geq 5, a \equiv 3 \bmod 8 \end{cases}$$

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Sketch of case $a \equiv 1 \bmod 8$ for $\bar{D}_2(a;n)$:

- Claim: $k^2 + 1 + 8m$ is a square mod $2^t \Leftrightarrow k = 4\ell$ for some $\ell \in \mathbb{Z}$.

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- " $\Leftarrow$ " Reduce mod 8, so $(4\ell)^2+1+8m \equiv 1 \bmod 8$. Use Proposition 4.
- Then, by Theorem 2, $\#\{k : k^2+1+8m \text{ is a square mod } 2^t, 0 \le k < 2^{t-1}\}$ $= \#\{4\ell : 0 \le \ell < 2^{t-3}\}$.

Explicit Formulas contd.

In the case when p is an odd prime, for $t \geq 3$,

$$\#\bar{S}_2(a;p^t) = \begin{cases} \frac{(p-3)p^{t-1}}{2} + \frac{p^{t-1}}{p+1} + \frac{3}{2} + \frac{(-1)^{t-3}(p-1)}{2(p+1)} & \left(\frac{a}{p}\right) = 1 \\ \frac{\phi(p^t)}{2} & \left(\frac{a}{p}\right) = -1 \end{cases}$$

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- Proof of 1 follows from cardinality formulas.
- Proof of 2 and 3 follow from [EKSY]. Only need to look at $p \equiv 3 \bmod 4$.

d-dimensional Modular Hyperbolas

Lemma

Let $F(x_1, \cdots, x_k) \equiv 0 \bmod p^t$ where $p > 2$. Then, the number of solutions is

$$S = \frac{1}{p^t} \sum_{u, x_1, \cdots, x_k} e^{2\pi i u F(x_1, \cdots, x_k)/p^t}$$

where $0 \leq u, x_1, \cdots, x_k < p^t$.

Sketch of proof:
Note that

$$\frac{1}{p^t} \sum_{u=0}^{p^t - 1} e^{2\pi i u x/p^t} = \begin{cases} 1 & x \equiv 0 \bmod p^t \\ 0 & x \not\equiv 0 \bmod p^t. \end{cases}$$

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Each solution contributes 1 to S, while a non-solution doesn't contribute to S.

Cardinality

Theorem

If $2, 3$ and $5 \nmid n$ and $d > 2$, the cardinality of $\bar{S}_d(m; a; n)$ is n.

Proof sketch:

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- Idea: show there are many solutions.

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- Change of variables:
  $$p^t + \frac{1}{p^t}\sum_{u=1}^{p^t-1}\sum_{y(p^t)} e^{(-2\pi i u b+2\pi i(a+y)^2 4^{-1}\varepsilon u y)/p^t}\sum_{x=0}^{p^t-1} e^{2\pi i x^2 u y\varepsilon/p^t}.$$

Cardinality Proof

- Note, by generalized Gauss sums
$\sum_{x=0}^{p^t-1} e^{2\pi i x^2 u y \varepsilon / p^t} = c\sqrt{p^t}$ where c has magnitude 1.

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$$p^t + d\frac{\sqrt{p^t}\sqrt{p^t}}{p^t}\sum_{y=0}^{p^t-1}\left(\frac{(a+y)^2 4^{-1}\varepsilon y^2 - \varepsilon b y}{p^t}\right)$$

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where the magnitude of d is 1.
- Main term is p^t. Rest of sum is bounded in magnitude by $p^t - 1$.

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- Behavior is the same for $\bar{S}_d(m; a; n)$ where $d > 2$.
- For $d = 2$, behavior is varied, so ratios lead to interesting behavior.

Future and Ongoing Research

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- Pick elements randomly with probability depending on the dimension of the modular hyperbola.
- Ratios for $H_2(a; n)$ where a is not a square mod n.

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