Futurama Theorem and Products of Distinct Transpositions

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Keeler's Method

Let $P = C_1 \cdots C_r$, where the C_i are disjoint cycles. Let $C_1 = (a_1 \ldots a_k)$, and define

$$\sigma_1 = (xa_1)(xa_2)\cdots(xa_{k-1}) \cdot (ya_k)(xa_k)(ya_1). \tag{1}$$

Then $\sigma_1 C_1 = (xy)$, where x, y are outsiders. For each C_i, we have analogous products σ_i for which $\sigma_i C_i = (xy)$. Taking

$$\sigma = \begin{cases} (xy)\sigma_1\sigma_2\cdots\sigma_r, & \text{if } r \text{ is odd} \\ \sigma_1\sigma_2\cdots\sigma_r, & \text{if } r \text{ is even,} \end{cases} \tag{2}$$

we find that σ **undoes** P. Let n be the number of entries in P. By (1) and (2), the number of factors in Keeler's σ is either $\mathbf{n+2r+1}$ or $\mathbf{n+2r}$ according as r is odd or even.

Our Optimal Refinement of Futurama/Keeler's Theorem

Theorem

Let $P = C_1 \cdots C_r$ be a product of r disjoint cycles in S_n, where n is the number of entries in P. Then P can be undone by a product λ of $\mathbf{n+r+2}$ *distinct transpositions in S_{n+2} each containing at least one of the outside entries $x = n+1$, $y = n+2$. Moreover, this result is* **best possible** *in the sense that $n+r+2$ cannot be replaced by a smaller number.*

Our Best Possible Algorithm

Recall $C_1 = (a_1 \ldots a_k)$. Corresponding to the cycle C_1, define

$$G_1(x) = (a_1x)(a_2x)\cdots(a_kx), \qquad F_1(x) = (a_1x).$$

Corresponding to each cycle C_i, $i = 1, \ldots, r$, define $G_i(x)$ and $F_i(x)$ analogously. Set

$$\lambda = (xy) \cdot G_r(x) \cdots G_2(x) \cdot (a_kx)G_1(y)(a_1x) \cdot F_2(y) \cdots F_r(y).$$

It is readily checked that λ undoes P and that λ is a product of $\mathbf{n+r+2}$ distinct transpositions in S_{n+2} each containing at least one of the outside entries x, y.

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Two Lemmas

Lemma (1)

Let $2 \leq k \leq n$. Then no k-cycle in S_n can be a product of fewer than $k-1$ transpositions.

Lemma (2)

Let $2 \leq k \leq n$. Suppose that $(a_1 \cdots a_k) \in S_n$ equals a product P of exactly $k-1$ transpositions in S_n. Then every entry in the product P is one of the a_i.

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4. Move (13) to the far right in σ. New $\sigma = \tilde{a}\tilde{b}\tilde{c}\tilde{d}\tilde{e}\tilde{f}(13)$.

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5. $\tilde{a}\tilde{b}\tilde{c}\tilde{d}\tilde{e}\tilde{f}(13)(13) = (9876543)(21)(13) = (987654321)$.
6. $6 < (9 - 1)$, which contradicts Lemma 1.

□

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Theorem

For $n > 1$, let $\mathbf{P} = (\mathbf{12})(\mathbf{34})\cdots(\mathbf{2n-1}, \mathbf{2n})$, a product of n disjoint transpositions in S_{2n}. Then P can be undone by a product of $2n+1$ or $2n$ distinct transpositions according as n is odd or even. This result is **best possible**.

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Number of switches: $2n + 1 = 2(5) + 1 = 11$

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Thank YOU for listening! :)