Phase Transitions in Generalized Sumsets

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Outline

Introduction

Statement

A finite set of integers, $|A|$ its size. Form

- Sumset: $A + A = \{a_i + a_j : a_j, a_j \in A\}$.
- Difference set: $A - A = \{a_i - a_j : a_j, a_j \in A\}$.

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Definition

We say A is difference dominated if $|A - A| > |A + A|$, balanced if $|A - A| = |A + A|$ and sum dominated (or an MSTD set) if $|A + A| > |A - A|$.

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- What happens when we increase the number of summands?
- What happens if we let the probability of choosing elements decays with N?
- What happens if we take subsets of non-abelian groups?

Past Results

- Martin and O'Bryant, 2006: Positive percentage of sets are MSTD when sets chosen with uniform probability.

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- Iyer, Lazarev, Miller, Zhang, 2011: Generalized results above to an arbitrary number of summands.

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- Found critical value of $\delta = \frac{1}{2}$ for probability $p(N) = cN^{-\delta}$, $\delta \in (0, 1)$. δ corresponds to the order of the number of repeated elements in the sumset.
- We call the critical value the phase transition because it is the value at which the order of the number of repeated elements is as large as the number of distinct elements.

Generalized Sumsets

Definition

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We want to study the size of this set as a function of s,d, and δ for probability $p(N) = cN^{-\delta}$.

Our goal: Extend the results of Hegarty-Miller to the case of Generalized Sumsets and determine where the phase transition occurs for $h > 2$.

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- Can we say anything about the relationship between δ and the ratio $\frac{|A_{s_1,d_1}|}{|A_{s_2,d_2}|}$ for $s_1 + d_1 = s_2 + d_2 = h$?
- What is the critical value as a function of δ?

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These three cases correspond to the speed at which the probability of choosing elements decays to 0.

Fast Decay

- For $\delta > \frac{h-1}{h}$, the set with more differences is larger 100% of the time.
- Ratio is a function of $\binom{h}{d}$.
- Results rely on the scarcity of elements chosen to be in A.

Proof I

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- Define $Y = \sum_{a,b} Y_{a,b}$
- Bound the expected value and variance of Y.

Proof II

- Use Chebyshev's Inequality:

$$\Pr\left(|X-\mu| \geq k\sigma\right) \leq \frac{1}{k^2}$$

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- Conclude that almost all h-tuples generate a distinct number as $N \to \infty$.

- Using combinatorics, conclude that ratio is:
$\frac{|A_{s_1,d_1}|}{|A_{s_2,d_2}|} = \frac{\binom{h}{d_1}}{\binom{h}{d_2}} = \frac{s_2!d_2!}{s_1!d_1!}$.

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- For $A + A + A$, $g(x) = \sum (-1)^{k-1}\left(\frac{1}{k12^k} + \frac{c_k}{(-8)^k}\right) x^k$ with $c_k = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} (x^2 - 1)^k dx$.

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Slow Decay

- If $\delta < \frac{h-1}{h}$, an even more delicate argument is needed.
- Now the number of repeated elements are of a higher order.
- Martingale Machinery of Kim and Vu.

Non-Abelian Finite Groups

Some new Definitions

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- So the sumset becomes $S \cdot S = \{xy : x, y \in S\}$.
- While the sum-difference becomes $S \cdot S^{-1} = \{xy^{-1} : x, y \in S\}$.

Dihedral Group

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I	I	R_{90}	R_{180}	R_{270}	L_1	L_2	L_3	L_4
R_{90}	R_{90}	R_{180}	R_{270}	I	L_4	L_1	L_2	L_3
R_{180}	R_{180}	R_{270}	I	R_{90}	L_3	L_4	L_1	L_2
R_{270}	R_{270}	I	R_{90}	R_{180}	L_2	L_3	L_4	L_1
L_1	L_1	L_2	L_3	L_4	I	R_{90}	R_{180}	R_{270}
L_2	L_2	L_3	L_4	L_1	R_{270}	I	R_{90}	R_{180}
L_3	L_3	L_4	L_1	L_2	R_{180}	R_{270}	I	R_{90}
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R_{270}	R_{270}	I	R_{90}	R_{180}	L_2	L_3	L_4	L_1
L_1	L_1	L_2	L_3	L_4	I	R_{90}	R_{180}	R_{270}
L_2	L_2	L_3	L_4	L_1	R_{270}	I	R_{90}	R_{180}
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- Note: at least half the elements in D_{2n} are of order 2.

Theorem

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If we let S be a random subset of D_{2n} (if $\alpha \in D_{2n}$ then $\mathbb{P}(\alpha \in S) = 1/2$) then

$$\lim_{n\to\infty} \mathbb{P}(|S \cdot S| = |S \cdot S^{-1}|) = 1.$$

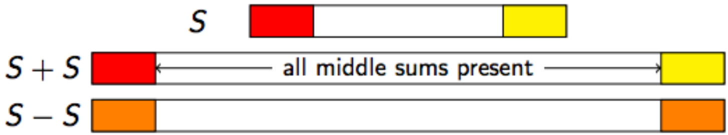
It is also true that

$$\lim_{n\to\infty} \mathbb{P}(|S \cdot S| = |S \cdot S^{-1}| = 2n) = 1.$$

We compute this instead, as it serve as a sufficient lower bound.

Intuition

- Key Idea: In the $\mathbb{Z}$ case, fringe matters most, middle sums and differences are present with high probability



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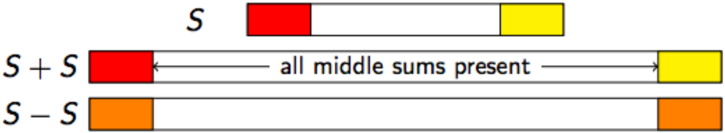

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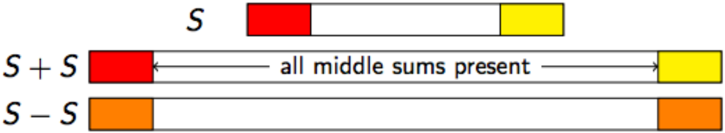

- *If we choose the "fringe" of S cleverly, the middle of S will become largely irrelevant. - Martin O'Bryant 2007*

- In $\mathbb{Z}/n\mathbb{Z}$ there is no fringe. So the "largely irrelevant" is the only thing that can be relevant.

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- For now we can ignore $R - R, -R + F$.
- We use that both $F + F$ and $R + F$ are in $S \cdot S$ and $S \cdot S^{-1}$ to compute lower bounds.

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- so elements in $R + R$ look like a^{x+y}, while elements in $F + F$ look like $a^{x-y} b$.
- This observation allows us to look at powers of the elements as an cyclic group instead.

Flips and Rotations

- If we let R^* and F^* be random subsets of $\mathbb{Z}/n\mathbb{Z}$ then we have the following:

$$\begin{aligned}\mathbb{P}(|S\cdot S^{-1}| = |S\cdot S|) &\geq \mathbb{P}(|S\cdot S^{-1}| = 2n \text{ and } |S\cdot S| = 2n)\\ &\geq \mathbb{P}(|S\cdot S^{-1}| = 2n)\mathbb{P}(S\cdot S = 2n)\\ &\geq \mathbb{P}(|S\cdot S^{-1}| = 2n)^2\\ &= \mathbb{P}(|F^* - F^*| = n \;\&\; |F^* + R^*| = n)^2\end{aligned}$$

Probability of Missing Elements

- We now proceed by computing the probability that an element is *not* in the desired set.

Lemma (Number of Missing Flips)

$\mathbb{P}(k \notin F^* + R^*) = O((3/4)^n)$

- This follows immediately from the number of ways one can add numbers in $\mathbb{Z}/n\mathbb{Z}$ to equal k.

Lemma

Lemma (Number of Missing Rotations)

$\mathbb{P}(k \notin F^* - F^*) = \frac{f(n/d)^d}{2^n} \leq (\varphi/2)^n$ where $gcd(k, n) = d$ and $f(n) = F(n+1) + F(n-1)$ where $F(n)$ is the nth Fibonacci number

The proof does not follow as immediately as it requires some combinatorics.

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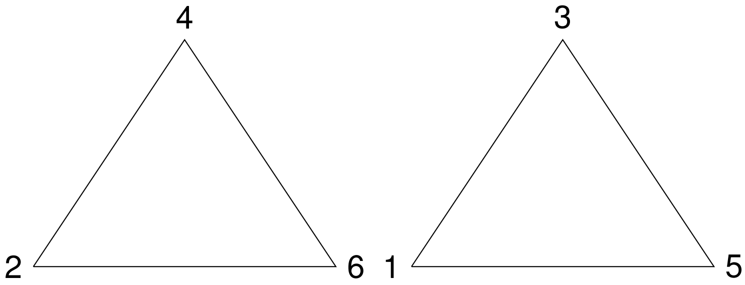

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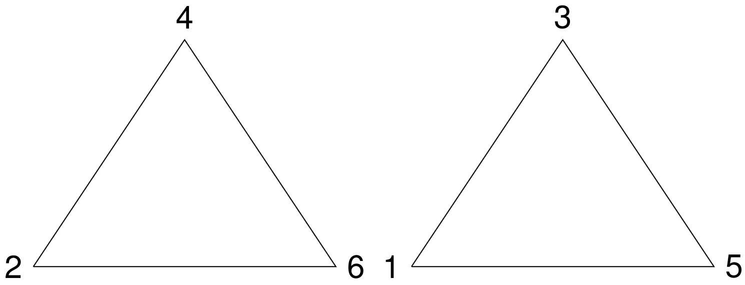
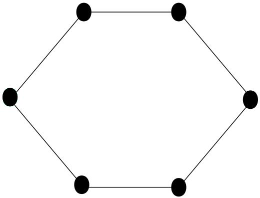


- Note that we get $\gcd(2,6) = 2$ number of polygons and they each have $6/\gcd(2,6) = 3$ vertices.

So where does Fibonacci come from?

- So the problem is thus reduced to counting the number of ways how the vertices of a n polygon can be colored either red or blue such that there are no 2 red vertices next to each other.



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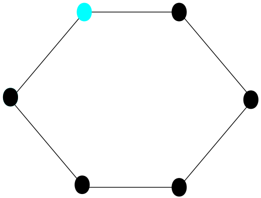

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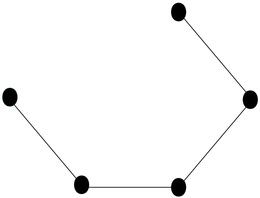

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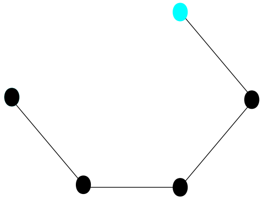

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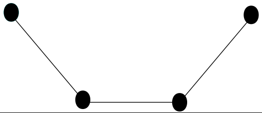

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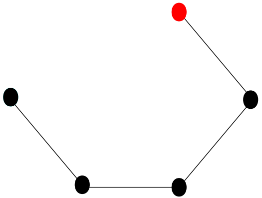

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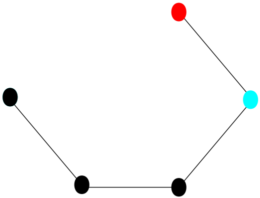

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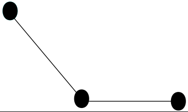

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- So by our two lemmas we have that as n goes to infinity,

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- Since this is the lower bound, we are done.

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Theorem (Semi-Direct Products)

For the group $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/m\mathbb{Z}$, if either n or m go to infinity then, $\mathbb{P}(|S \cdot S| = |S \cdot S^{-1}|) = 1$.

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Theorem ([Abelian Groups)

As the size of an abelian group approaches infinity, then $\mathbb{P}(|S \cdot S| = |S \cdot S^{-1}|) = 1$.

Ping Pong

Theorem (Free Group)

If we let $\langle a, b \rangle_l$ be all words up to length l and $S \subseteq \langle a, b \rangle_l$ then as l goes to infinity we have that:

$$\mathbb{P}(|S \cdot S| \geq |S \cdot S^{-1}|) = 1$$


TAKE THAT
SUM DIFFERENCE!
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