Distribution of the Longest Gap in Positive Linear Recurrence Sequences

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Introduction

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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Gaps: Our object of study

Instead of looking at the number of summands, we study the spacings between them, as follows:

Definition: Gaps

If $x \in [F_n, F_{n+1})$ has Zeckendorf decomposition $x = F_n + F_{n-g_1} + F_{n-g_2} + \cdots + F_{n-g_k}$, we define the *gaps* in its decomposition to be $\{g_1, g_1 - g_2, \cdots, g_{k-1} - g_k\}$.

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Example:

- $2012 = F_{16} + F_{13} + F_8 + F_3 + F_1$.
- Gaps of length 3, 5, 5, and 2.

Previous Results (Miller-Beckwith)

Theorem (Zeckendorf Gap Distribution)

For Zeckendorf decompositions, $P(k) = \frac{1}{\phi^k}$ *for* $k \geq 2$*, with* $\phi = \frac{1+\sqrt{5}}{2}$ *the golden mean.*

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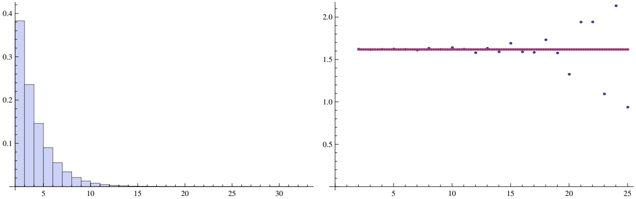

Figure: Distribution of gaps in $[F_{1000}, F_{1001})$; $F_{1000} \approx 10^{208}$.

Our Problem:

Given a random number x in the interval $[F_n, F_{n+1})$, what is the probability that x has longest gap equal to r?

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- Philadelphia International Airport

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First we'll review what we need to know.

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Using geometric series expansion, this is equal to

$$\left(\frac{1 - x^n}{1 - x}\right)^k.$$

Partial Fraction Expansion

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Using the method of partial fractions, we can write this as:

$$f(x) = \frac{c_1}{x+\phi} + \frac{c_2}{x-1/\phi},$$

where $-\phi$ and $1/\phi$ are the roots of $1-x-x^2$, and c_1 and c_2 are determined by algebra to be $c_1 = c_2 = \sqrt{5}$.

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And by Binet's well known formula for the Fibonacci numbers, we see that:

$$f(x) = \sum_{i=0}^{\infty} F_i x^i.$$

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Results

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Pick x randomly from the interval $[F_n, F_{n+1})$. We prove explicitly the cumulative distribution of x's longest gap.

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Theorem

Let $r = \phi^2/(\phi^2+1)$. Define f as $f(n,u) = \log rn / \log\phi\ + u$ for some fixed $u \in \mathbb{R}$. Then, as $n \to \infty$, the probability that $x \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$\mathbb{P}(L(x) \leq f(n,u)) = e^{e^{(1-u)\log\phi + \{f(n)\}}}$$

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Immediate Corollary: If $f(n,u)$ grows any **slower** or ***faster*** than $\log n/\log\phi$, then $\mathbb{P}(L(x) \leq f(n))$ goes to **0** or ***1*** respectively.

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The mean is given by

$$\mu = \int_{-\infty}^{\infty} u \frac{\mathrm{d}}{\mathrm{d}u} P(u) \mathrm{d}u.$$

The variance follows similarly.

Mean and Variance

So the mean is about

$$\mu = \frac{\log\left(\frac{\phi^2}{\phi^2+1}\right)}{\log\phi} + \int_{-\infty}^{\infty} e^{-e^{(1-u)\log\phi}} e^{(1-u)\log\phi} \log\phi \, du.$$

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In the continuous approximation, the mean is

$$\frac{\log\left(\frac{\phi^2}{\phi^2+1} n\right)}{\log\phi} - \gamma,$$

where γ is the Euler- Mascheroni constant.

Fibonacci Case Generating Function

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$G_{n,k,f}$ is the coefficient of x^{n+1} for the generating function

$$\frac{1}{1-x}\left[\sum_{j=2}^{f(n)-1} x^j\right]^{k-1}$$

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- Each gap is $< f(n)$.

The gaps uniquely identify y because of Zeckendorf's Theorem!

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For fixed k, this is surprisingly hard to analyze. We only care about the sum over all k.

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Let's work with this (use **partial fractions** and **Rouché**) to find the CDF.

Partial Fractions pt 2

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We can take the $n+1$st coefficient of this expansion to find the number of y with gaps less than $f(n)$.

Partial Fractions pt 3

Divide the **number** of $y \in [F_n, F_{n+1})$ with longest gap $< f(n)$, **by** the total number of y, which is $F_{n+1} - F_n \sim \frac{1}{\sqrt{5}}\phi^n$.

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Theorem

The proportion of $y \in [F_n, F_{n+1})$ with $L(x) < f(n)$ is exactly

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Now, we find out about the roots of $x^f - x^2 - x + 1$.

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For $f \in \mathbb{N}$ and $f \geq 4$, the polynomial $p_f(z) = z^f - z^2 - z + 1$ has exactly one root z_f with $|z_f| < .9$. Further, $z_f \in \mathbb{R}$ and $z_f = \frac{1}{\phi} + \left|\frac{z_f^f}{z_f + \phi}\right|$, so as $f \to \infty$, z_f converges to $\frac{1}{\phi}$.

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And we only care about the **small roots.**

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Theorem

*If $\lim_{n\to\infty} f(n) = \infty$, the **proportion** of y with $L(y) < f(n)$ is, as $n \to \infty$*

$$\lim_{n\to\infty} (\phi z_f)^{-n} = \lim_{n\to\infty} \left(1 + \left|\frac{\phi z_f^{f(n)}}{\phi + z_f}\right|\right)^{-n}.$$

And if $f(n)$ is bounded, then $P_f = 0$.

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In the limit, if $\lim_{n\to\infty} f(n) = \infty$ *then*

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- Take logarithms

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And if $f(n)$ *is bounded, then* $P_f = 0$.

We're almost there!:

- Take logarithms
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We're almost there!:

- Take logarithms
- Taylor expand
- Use that $z_f \to 1/\phi$ as $n \to \infty$
- Re-exponentiate

Cumulative Distribution Function

After some technical details, we get our long awaited theorem!

Theorem

Let $r = \phi^2/(\phi+1)$*. Define* f : *as* $f(n) = \log rn/\log\phi\ + u$ *for some fixed* $u \in \mathbb{R}$*. Then, as* $n \to \infty$*, the probability that* $x \in [F_n, F_{n+1})$ *has longest gap less than* $f(n)$ *is*

$$\mathbb{P}(L(x) < f(n)) = e^{-e^{u\log\phi}+\{f\}}$$

Conclusions and Future Work

Positive Linear Recurrence Sequences

This method can be greatly generalized to **Positive Linear Recurrence Sequences** ie: linear recurrences with non-negative coefficients. WLOG:

$$H_{n+1} = c_1 H_{n-(t_1=0)} + c_2 H_{n-t_2} + \cdots + c_L H_{n-t_L}.$$

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Theorem (Zeckendorf's Theorem for *PLRS* recurrences)

Any $b \in \mathbb{N}$ has a unique ***legal*** *decomposition into sums of H_n,*
$b = a_1 H_{i_1} + \cdots + a_{i_k} H_{i_k}$.

Here **legal** reduces to non-adjacency of summands in the Fibonacci case.

Generating Function, pt n

The **number** of $b \in [H_n, H_{n+1})$, with longest gap $< f$ is the coefficient of x^{n-s} in the generating function:

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A geometric series!

Some More Definitions

Let λ_i be the eigenvalues of the recurrence, and p_i their coefficients. Define

$$\mathcal{G}(x) = \prod_{i=2}^{L}\left(x - \frac{1}{\lambda_i}\right)$$

$$\mathcal{P}(x) = (c_1 - 1)x^{t_1} + c_2 x^{t_2} + \cdots + c_L x^{t_L},$$

$$\mathcal{R}(x) = c_1 x^{t_1} + c_2 x^{t_2} + \cdots + (c_L - 1)x^{t_L}.$$

and

$$\mathcal{M}(x) = 1 - c_1 x - c_2 x^{t_2+1} - \cdots - c_L x^{t_L+1}$$

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There is again a critical root, $z_f \to 1/\lambda_1$ exponentially as $f \to \infty$.

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$$\mathbb{P}(L(x) < f) = \frac{-\mathcal{P}(z_f)\ /\ (p_1\lambda_1 - p_1)}{z_f\mathcal{M}'(z_f) + f\ z_f^f\ \mathcal{R}(z_f) + z_f^{f+1}\mathcal{R}'(z_f)}\left(\frac{1}{z_f\lambda_1}\right)^n$$

$$+H(n, f)$$

where there exists ϵ *with* $1/\lambda_1 < \epsilon < 1$, *such that* $H_{n,f} \ll f\epsilon^n$

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Our techniques handle this!

References

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