Spectral Statistics of Non-Hermitian Matrix Ensembles

Ryan Chen, Eric Winsor
rcchen@princeton.edu, rcwnsr@umich.edu
with Yujin Kim, Jared Lichtman, Alina Shubina, and Shannon Sweitzer
Advisor: Steven J. Miller
Introduction
Random Matrix Ensembles

A random matrix ensemble is a collection of matrices, with some probability assigned to each matrix.

For us, we will consider ensembles with matrix probabilities given by a product of entry probabilities:

\[ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} \]

Fix \( p \), define

\[ \text{Prob}(A) = \prod_{1 \leq i, j \leq N} p(a_{ij}). \]
Random Matrix Ensembles - Examples

Examples:

- Real symmetric matrices
- Hermitian matrices
- Real asymmetric matrices
- Complex symmetric/asymmetric matrices

*Real eigenvalues*: Real symmetric matrices, Hermitian matrices

*Complex eigenvalues*: Real asymmetric matrices, Complex symmetric/asymmetric matrices
The basic question: Given a random matrix ensemble, what does the distribution of eigenvalues look like as we send the matrix dimension $N \to \infty$?

Wigner’s Semi-Circle Law (Wigner 1958)

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all $A$, as $N \to \infty$

$$\mu_{A,N}(x) \to \begin{cases} \frac{2}{\pi} \sqrt{1 - x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$
The \textit{kth-moments} $M_k$ of a probability distribution $\mu$ are given by

\[ M_k := \int_{-\infty}^{\infty} x^k \, d\mu. \]

Let $A$ be an $N \times N$ matrix with eigenvalues $\lambda_i$.

\begin{itemize}
  \item $M_k(N) := \frac{1}{N} \cdot \mathbb{E} \left[ \sum_{i=1}^{N} \left( \frac{\lambda_i}{\sqrt{N}} \right)^k \right]$
  \item $\sum_{i=1}^{N} \lambda_i^k = \text{Trace}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1i_2} \cdots a_{i_ki_1}$
\end{itemize}
Numerical Support

(Rescaled) 500 Matrices (Gaussian) $250 \times 250$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

The even moments are given by the Catalan numbers.
Ensembles with Complex Eigenvalues
Moment analysis has its limitations - for example, in the complex plane, every radially symmetric distribution has all moments zero!

Circular Law - Complex Asymmetric (Tao and Vu, 2008)

Consider the ensemble of $N \times N$ complex asymmetric random matrices, with iidrv complex random variables with mean 0 and variance 1. Then, after normalizing by $\sqrt{N}$, the eigenvalue distribution converges to the uniform distribution on the unit disk.
Numerical Support

(Rescaled) 500 Matrices (Gaussian) $250 \times 250$
We will study complex symmetric matrices. It turns out that, for sufficient conditions on the higher moments of the base distribution, we also obtain a circular law:

**Circular Law - Complex Symmetric (SMALL 2017)**

Consider the ensemble of $N \times N$ complex symmetric random matrices, with iidrv complex random variables with mean 0 and variance 1. Then, after normalizing by $\sqrt{N}$, the eigenvalue distribution converges to the uniform distribution on the unit disk.
Complex Symmetric Matrices - Singular Values
Singular Values

- The *singular values* of a matrix $A$ are defined as the square roots of the eigenvalues of $A^*A$.
- Since $A^*A$ is Hermitian, these eigenvalues will be real.
- Furthermore, $A^*A$ positive definite implies all the eigenvalues are nonnegative.

**Quarter Circular Law - Complex Asymmetric (Marchenko and Pastur, 1967)**

Consider the ensemble of $N \times N$ complex asymmetric random matrices, with iidrv complex random variables with mean 0 and variance 1. Then, after normalizing by $\sqrt{N}$, the singular value distribution converges to a quarter circle.
Numerical Support

(Rescaled) 500 Matrices (Gaussian) $250 \times 250$
It turns out that, for sufficient conditions on the higher moments of the base distribution, we also obtain a quarter circular law for complex symmetric matrices:

Quarter Circular Law - Complex Symmetric (SMALL 2017)

Consider the ensemble of $N \times N$ complex symmetric random matrices, with iidrv complex random variables with mean 0 and variance 1. Then, after normalizing by $\sqrt{N}$, the singular value distribution converges to a quarter circular law.
## Joint Density Functions (Gaussian Distributions)

### Joint Density - Real Asymmetric Singular Values
(James 1960)

\[
\rho_N(z_1, \ldots, z_N) = c_N \prod_{1 \leq i < j \leq N} |z_j^2 - z_i^2| \prod_{1 \leq i \leq N} e^{-|z_i|^2}
\]

### Joint Density - Complex Asymmetric Singular Values
(James 1963)

\[
\rho_N(z_1, \ldots, z_N) = c_N \prod_{1 \leq i < j \leq N} |z_j^2 - z_i^2| \prod_{1 \leq i \leq N} |z_i| \prod_{1 \leq i \leq N} e^{-|z_i|^2}
\]
Joint Density - Complex Symmetric Singular Values
(SMALL 2017)

\[ \rho_N(z_1, \ldots, z_N) = c_N \prod_{1 \leq i < j \leq N} |z_j^2 - z_i^2| \prod_{1 \leq i \leq N} |z_i| \prod_{1 \leq i \leq N} e^{-|z_i|^2} \]
Joint Densities - 2-tuples of singular values of $2 \times 2$ matrices

**Figure:** Real and Complex Asymmetric
Joint Densities - 2-tuples of singular values of $2 \times 2$ matrices

Figure: Complex Symmetric
Structured Families - Complex Checkerboard Matrices
A complex symmetric \((k, w)\)-checkerboard random matrix is a complex symmetric matrix made up of \(k \times k\) blocks of the form

\[
B = \begin{pmatrix}
  w & * & \cdots & *
  \\
  * & w & \cdots & *
  \\
  \vdots & \vdots & \ddots & \vdots
  \\
  * & * & \cdots & w
\end{pmatrix}
\]

where \(w\) is some fixed constant and each \(*\) represents a random variable.
The limiting singular value distribution for the complex symmetric checkerboard ensemble shows split limiting behavior:

1. A "bulk" containing most of the mass following the Quarter Circle Law.
2. A "blip" of density $k/N$ with mean close to $Nw/k$ whose distribution is that of the hollow GOE ensemble (random real symmetric matrices with 0 on the diagonal).
(Rescaled) 2000 Complex Symmetric (3,1)-Checkerboard Matrices
100 × 100
Proof Outline - Checkerboard Singular Value Bulk

- Reduce to case of $w = 0$ by using a matrix perturbation argument.
- Apply eigenvalue-trace and method of moments.
- Use combinatorial argument similar to Wigner Semicircle Law proof.
Harder: Define the **empirical blip square singular spectral measure** (EBSSSM) for a matrix $A$ to be

$$
\mu_{A,N}^{s^2} := \frac{1}{k} \sum_{\sigma} f_n(N) \left( \frac{k^2 \sigma}{N^2} \right) \delta \left( x - \frac{1}{N} \left( \sigma - \frac{N^2}{k^2} \right) \right)
$$

where $\sigma$ ranges over singular values of $A$,

$$
f_n(x) = x^{2n} (x - 2)^{2n}
$$

and $n(N)$ is some slow growing function.
We apply the method of moments to show that this distribution converges to the square of the hollow GOE distribution in the case of the complex symmetric checkerboard ensemble.
Eigenvalue Distribution for Complex Symmetric Checkerboard Ensemble (SMALL 2017)

The limiting singular value distribution for the complex symmetric checkerboard ensemble shows split limiting behavior:

1. A "bulk" containing most of the mass following the Circular Law.
2. A "blip" of density \( k/N \) with mean close to \( Nw/k \).
(Rescaled) 500 Complex Symmetric (3,1)-Checkerboard Matrices
300 × 300
Use knowledge of singular values.

**Fact**

The largest singular value of a complex symmetric matrix $A$ has magnitude greater than or equal to that of the largest eigenvalue of $A$.

The absolute values of the eigenvalues must be bounded above by $N/k + O(1)$. Look at eigenvalues of $kN^{-1}A$. 
Checkboard Eigenvalue Blip

Eigenvalues of $\frac{k}{N} A$: Contained within disk of radius $1 + O\left(\frac{1}{N}\right)$.

- Show that mean of eigenvalue distribution is $\frac{k}{N} + O\left(N^{-2}\right)$.
- Show that the only eigenvalues contributing to the mean must be on the boundary of the disk.
- Conclude that there must be $\frac{k}{N}$ eigenvalues at 1.
Future Work

- Different placement of $w$’s and different block shapes.
- Complex asymmetric and combinations of checkerboarding with other ensembles.
- Allow the block size to grow as a function of $N$. 
References


Thank you!