# Variance of Gaussian Primes Across Sectors and the Hecke L-functions Ratios Conjecture

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## Introduction

A Gaussian prime is a prime element in the ring of Gaussian integers $\mathbb{Z}[i]$. Since they are located in the complex plane, interesting questions regarding their geometric properties naturally arise. We associate to each Gaussian prime $a+bi$ an angle whose tangent is $b/a$. Hecke showed in 1919 that these angles are uniformly distributed in the plane, and Kubilius proved uniform distribution in somewhat short arcs in 1950. Motivated by a random matrix theory model and a function field analogue, Rudnick and Waxman gave a conjecture for the variance of such angles across short arcs. We apply the $L$-functions ratios conjecture to a family of Hecke $L$-functions to derive a formula which computes the one-level density and variance of Gaussian primes across short arcs.

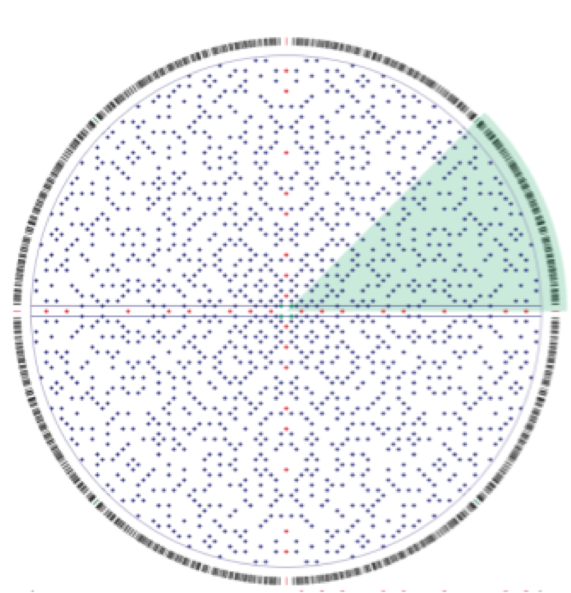

Figure 1: Gaussian primes uniformly distributed in the plane
https://illustratedtheoryofnumbers.wordpress.com/tag/primes/

## The Conjecture

It has been observed that the statistics for classical random matrix ensembles match the statistics for zeros of various families of $L$-functions, and that the characteristic polynomials for such matrix ensembles can model the $L$-functions. A natural extension is to use random matrices to model ratios of products of $L$-functions. However, the ratios conjecture captures the subtle arithmetic properties found in the correction factor, thus providing an even closer model.

## The Ratios Recipe

We wish to apply the ratios conjecture to

$$\frac{1}{K}\sum_{k\leq K}\frac{L_K(1/2+\alpha)L_K(1/2+\beta)}{L_K(1/2+\gamma)L_K(1/2+\delta)}.$$

1. We replace all $L$-functions in the numerator by their approximate functional equation. In our case, the approximate functional equation is given by

$$L_k(s)=\sum_n\frac{A(n)}{n^s}+\pi^{2s-1}\frac{\Gamma(1-s+|2k|)}{\Gamma(s+|2k|)}.$$

We replace $L$-functions in the denominator using the equation

$$\frac{1}{L_k(s)}=\sum_n\frac{U(n)}{n^s}.$$

We then multiply out the remaining expression.

2. We use Euler products to rewrite each term in the expanded expression as a product over primes, and then replace the values $A(p^m)A(p^n)U(p^h)U(p^l)$ with their averages over the family.

3. We now pull out terms that contribute zeros and poles to the products. In particular, these are Zeta functions and $L$-functions. We are then left with something of the form

$$\frac{A}{Y}(\alpha,\beta,\gamma,\delta),$$

where $Y$ is composed of the pulled out Zeta and $L$-functions and $A$ is the remaining arithmetic term.

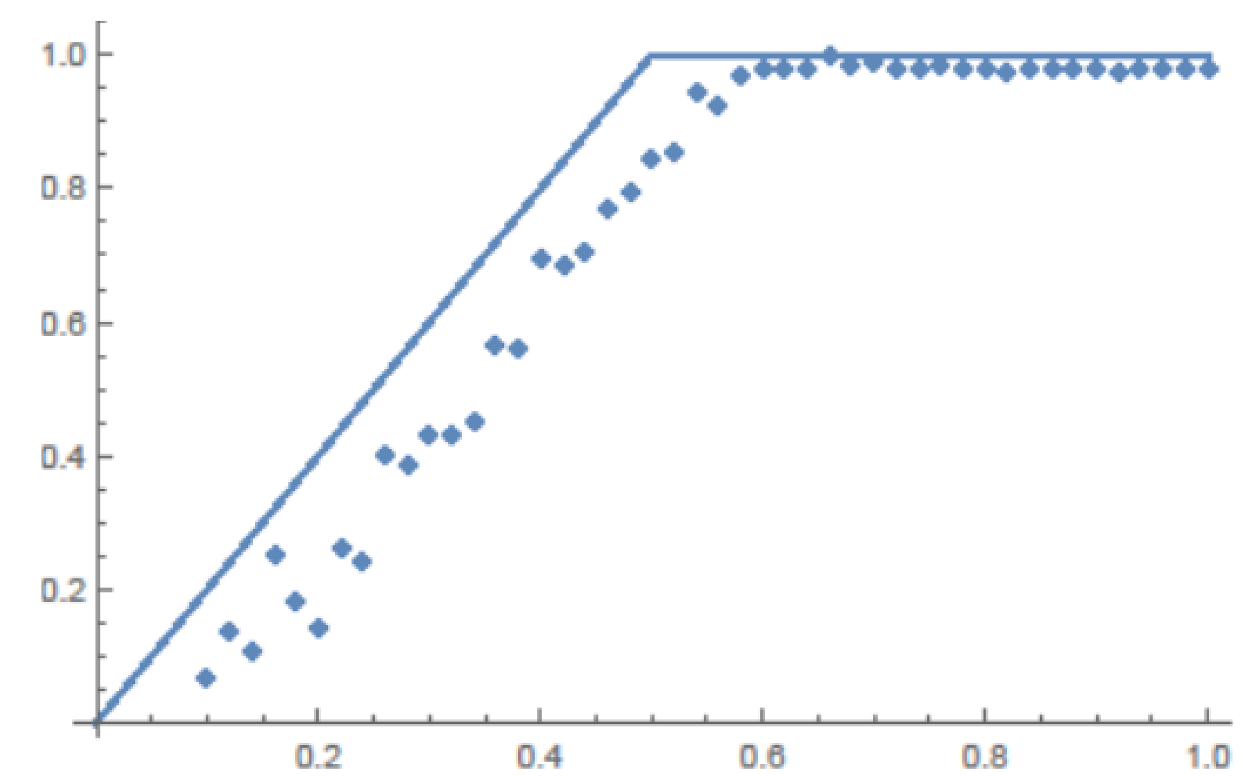


Figure 2: A plot of the ratio $\mathrm{Var}(NK,x)/E(NK,x)$ versus $\beta=\log K/\log N$, for $X\approx 108$ . The smooth line is $\min(1,2\beta)$.

## Application to the One-Level Density

We wish to compute $S_K(f)=\frac{1}{K}\sum_{k\leq K}\sum_{\gamma_k}f(\gamma_k)$, where $f$ is a nice even function, and $\frac{1}{2}+\gamma_k$ denotes the ordinate of a generic zero of $L_k(s)$ on the half-line. However, contour integration gives

$$\frac{1}{K}\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t-i(c-1/2))\sum_{k\leq K}\frac{L_k'}{L_k}(c+it)dt\quad\longrightarrow\quad\int_{-\infty}^{\infty}g(x)\Big(1-\frac{\sin(2\pi x)}{2\pi x}\Big)\,dx$$

where $s=c+it$. We apply the ratios conjecture to $L_k'(s)/L_k(s)$, and make the substitution $x=(2\pi/\log K)t$.

## Application to the Variance

We wish to compute

$$\mathrm{Var}(\psi_{K,X})=\frac{K^{\gamma-2}}{4\pi^2}\sum_{k\neq0}|\hat{f}(\frac{k}{K})|^2\int_{(c)}\int_{(c')}\frac{L_k'}{L_k}(\frac{1}{2}+\alpha)\frac{L_k'}{L_k}(\frac{1}{2}-\beta)\cdot\tilde{\Phi}(\frac{1}{2}+\alpha)\tilde{\Phi}(\frac{1}{2}+\beta)e^{i\lambda(\alpha-\beta)}d\alpha d\beta.$$

We apply the ratios conjecture to the expression

$$\frac{L_k(\frac{1}{2}+\alpha)L_k(\frac{1}{2}+\beta)}{L_k(\frac{1}{2}+\gamma)\overline{L}_k(\frac{1}{2}+\delta)},$$

and then take the second derivative with respect to $\alpha$ and $\beta$ to get a more manageable form of $\frac{L_k'}{L_k}(\frac{1}{2}+\alpha)\frac{L_k'}{L_k}(\frac{1}{2}-\beta)$.

## References

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