Introduction

A Gaussian prime is a prime element in the ring of Gaussian integers \( \mathbb{Z}[i] \). Since they are located in the complex plane, interesting questions regarding their geometric properties naturally arise. We associate to each Gaussian prime \( a + bi \) an angle whose tangent is \( b/a \). Hecke showed in 1919 that these angles are uniformly distributed in the plane, and Kubilius proved uniform distribution in somewhat short arcs in 1958. Motivated by a random matrix model and a function field analogue, Rudnick and Waxman gave a conjecture for the variance of such angles across short arcs. We apply the \( L \)-functions ratios conjecture to a family of Hecke \( L \)-functions to derive a formula which computes the one-level density and variance of Gaussian primes across short arcs.

The Conjecture

It has been observed that the statistics for classical random matrix ensembles match the statistics for zeros of various families of \( L \)-functions, and that the characteristic polynomials for such matrix ensembles can model the \( L \)-functions. A natural extension is to use random matrices to model ratios of products of \( L \)-functions. However, the ratios conjecture captures the subtle arithmetic properties found in the correction factor, thus providing an even closer model.

The Ratios Recipe

We wish to apply the ratios conjecture to

\[
\frac{1}{K} \sum_{k \leq K} L_k(1/2 + \alpha)L_k(1/2 + \beta)
\]

We replace all \( L \)-functions in the numerator by their approximate functional equation. In our case, the approximate functional equation is given by

\[
L_k(s) = \sum_{n \leq x} \frac{A(n)}{n^s} + \pi^{s-1/2} \Gamma(1-s) \left( \frac{|k|}{x} \right)^{s-1}.
\]

We replace \( L \)-functions in the denominator using the equation

\[
\frac{1}{L_k(s)} = \sum_{n \leq x} \frac{U(n)}{n^s}.
\]

We then multiply out the remaining expression.

We use Euler products to rewrite each term in the expanded expression as a product over primes, and then replace the values \( A(p^r)A(p^s)U(p^r)/U(p^s) \) with their averages over the family.

We now pull out terms that contribute zeros and poles to the products. In particular, these are Zeta functions and \( L \)-functions. We are then left with something of the form

\[
A \left( \frac{\alpha}{2}, \beta, \gamma, \delta \right)
\]

where \( Y \) is composed of the pulled out Zeta and \( L \)-functions and \( A \) is the remaining arithmetic term.

Application to the Variance

We wish to compute

\[
\text{Var}(\psi(x)) = \frac{K^{-1}}{4\pi^2} \sum_{h \neq 0} \left| \sum_{k \leq K} \frac{L_k^2}{L_k} \right|^2 \frac{1}{x^2} \int_0^1 \left( \phi(x)(1 + x) - \phi(x) \right) dx
\]

We apply the ratios conjecture to the expression

\[
\frac{L_k(1/2 + \alpha)L_k(1/2 + \beta)}{\left( \frac{\alpha}{2} + \beta \right) \left( \frac{\gamma}{2} + \beta \right) \left( \frac{\delta}{2} + \beta \right)}
\]

and then take the second derivative with respect to \( \alpha \) and \( \beta \) to get a more manageable form of \( \frac{\alpha}{2} \left( \frac{1}{2} - \beta \right) \).

References


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