Bounds for Vanishing of L-functions at the Central Point

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Introduction

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The distribution of prime numbers is closely linked to the distribution of the zeroes of the Riemann zeta function.

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An L-function is an analytic continuation of a function of the form

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where f is some mathematical object, and $a_f(n)$ is some sequence of coefficients encoding information about this object.

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In addition to this series formula, we often require L-functions to have other nice properties.

Properties of L-functions

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We usually want L-functions to have an Euler product form:

$$L(s, f) = \prod_{p \text{ prime}} L_p(s, f)^{-1}$$

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$$\Lambda(s,f) = \Lambda_\infty(s,f)\,L(s,f) = \Lambda(1-s,f)$$

Zeroes of L-functions

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We often want the same property to hold for zeroes of L-functions that we study. We call this the **Generalized Riemann Hypothesis**.

Zeroes and the n-Level Density

Zeroes of L Functions

Classically, statistical analysis of the zeroes of L-functions was insensitive to changes in finitely many zeroes. Many universal results for L-functions were found with these statistics. Concentrating on zeroes near the central point (the point $\frac{1}{2}$) offers the potential for new results:

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Birch and Swinnerton-Dyer Conjecture

The rank of the Mordell-Weil group of rational solutions of an elliptic curve is equal to the order of vanishing of the associated L-function at the central point.

1-Level Density

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Let $L(s, f)$ be an L-function. Let $\phi(x)$ be an even Schwartz (rapidly decaying) function. Let $\gamma_f^{(j)}$ denote the jth zero of L. Then the 1-level density is defined to be

$$D_{1,f}(\phi) = \sum_j \phi\left(L_f \gamma_f^{(j)}\right)$$

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$$D_{n,f}(\phi) = \sum_{\substack{j_1, \ldots, j_n \\ j_k \neq j_l}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

where L_f is a scaling parameter.

Connection with Random Matrix Theory - Katz Sarnak

The following conjecture bridges random matrix theory and zeroes of L-function:

Conjecture (Katz Sarnak)

The statistics of zeroes of L-functions are well modeled my random matrix ensembles. In particular, the n-level density of L-function zeroes is well modeled by random matrix theory.

Obtaining bounds on vanishing with n-level density

For a family $\bigcup \mathcal{F}_N$ of L-functions, we can write

$$\lim_{N\to\infty} \sum_{f\in\mathcal{F}_N} \sum_{\gamma} \mathrm{Rank}(f(0))\phi(0)$$

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$$= \int_{\mathbb{R}^n} \phi(x) W_{n,G}(x) dx.$$

The idea is simple: throw out the zeroes on the left hand side that aren't at the central point since ϕ is nonnegative everywhere. Thus:

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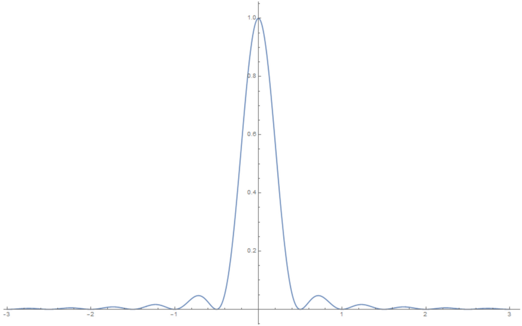
$$\lim_{N\to\infty} \mathrm{AvgRank}(\mathcal{F}_N) \leq \frac{\int_{\mathbb{R}^n} \phi(x) W_{n,G}(x) dx}{\phi(0)}$$

Optimal Test Functions : 1-level

Optimization in the 1-level

We can consider the test function $\phi(x) = \left(\frac{\sin 2\pi x}{2\pi x}\right)^2$



1-Level Optimization from ILS

The function $\phi(x) = \left(\frac{\sin 2\pi x}{2\pi x}\right)^2$ minimizes $\frac{\int_{-\infty}^{\infty} \phi(x) W(x) dx}{\phi(0)}$, for W corresponding to:

- Orthogonal
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and is almost minimal (but not quite) for

- Special Orthogonal (+,-,$*$)
- Symplectic

Fredholm Theory

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- Weight functions $\hat{W}(u)$ in the 1-level take the form $\delta(u) + m(u)$.
- The positivity condition on ϕ forces $\hat{\phi} = g * g$, via Ahiezer and Paley-Wiener.
- ILS rewrites the ratio $\frac{\int_{-\infty}^{\infty} \phi(x) W(x) dx}{\phi(0)} = \frac{\langle (I+K)g, g\rangle}{|\langle g, 1\rangle|^2}$ where $Kg = m * g$.
- Using Fredholm theory, we can reinterpret this as solving (I+K)g=1.

Better Bounds - Larger Support

One can ask for optimal test functions, given larger support for the Fourier transform.

These provide *conditional* results, since number theory does not have have agreement for large support of the Fourier transform.

Optimal Test Functions : 2-level

Fredholm Theory for the 2-level

We encounter significant obstacles with such methods in higher levels.

By Plancherel we can write:

$$\frac{\int_{\mathbb{R}^2} \phi_1(x_1)\phi_2(x_2)W(x)dx}{\phi_1(0)\phi_2(0)} = \frac{\int_{\mathbb{R}^2} \widehat{\phi}_1(x_1)\cdot\widehat{\phi}_2(x_2)\widehat{W}(x)dx}{\phi_1(0)\phi_2(0)}$$

It is of interest to study the special case $\phi := \phi_1 = \phi_2$.

2-level Unitary Example Calculation

The two level unitary weight takes the form

$$\widehat{W}_{2,U}(x_1, x_2) = \delta(x_1)\delta(x_2) - \delta(x_1 + x_2) \cdot (1 - |x_1|)I(x_1)$$

and we can obtain a similar-looking inner product ratio

$$\frac{\langle (I_0 + K')\hat{\phi}, \hat{\phi}\rangle}{\langle \hat{\phi}, 1\rangle^2}$$

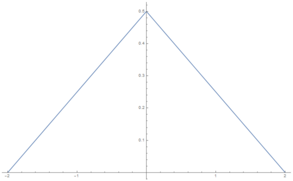
for some (now different) operators I_0 and K'.

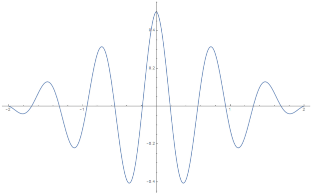
The point is that positivity of ϕ does not translate to the positivity of the Fourier transform, so we cannot apply Fredholm Theory as before.

Linear Combinations

Thus we restrict our attention to linear combinations of shifts of the original test function $\phi(x) = \left(\frac{\sin 2\pi x}{2\pi x}\right)^2$, still in the special case $\phi_1 = \phi_2 = \phi$.

The point is that shifts of the original test function do not change the support of the Fourier transform:





Consider test functions $\phi(x_1)\phi(x_2)$ with ϕ linear combinations of the form:

$$\phi(x) = \sum \alpha_i \left(\frac{\sin 2\pi(x + c_i)}{2\pi x} \right)^2$$

Theorem (SMALL 2017)

Test functions of the above yield best bounds when $c = 0$ and $\alpha_0 = 1$, for all families (Unitary, Orthogonal, Symplectic, SO +,-,$*$).

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Thank you!