Limiting Distributions of Generalized b-bin Zeckendorf Decompositions

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Introduction

- First, recall that the Fibonacci numbers are defined by the following recurrence relation: $F_{n+1} = F_n + F_{n-1}$.
Thus, we have $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$.

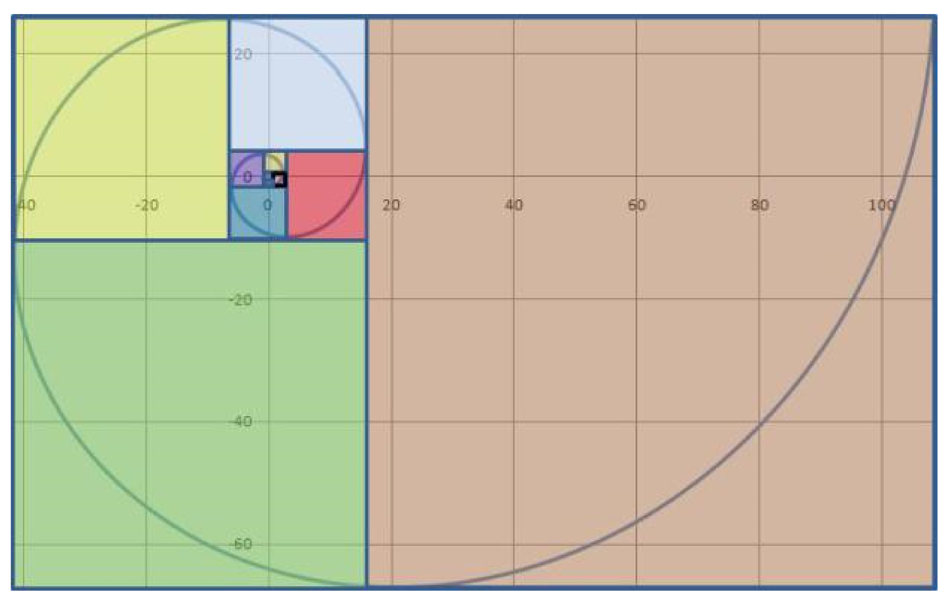

Figure 1: Geometrically, the Fibonaccis can be represented by a spiral

https://timwolverson.wordpress.com/2014/02/08/plot-a-fibonacci-spiral-in-excel/

- We can compute the generating function for the Fibonaccis: $g(x) = x/(1-x-x^2) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-x(\frac{1+\sqrt{5}}{2})} - \frac{1}{1-x(\frac{1-\sqrt{5}}{2})}\right)$.
- **Zeckendorf's Theorem.** Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers e.g. $2017 = 1597 + 377 + 34 + 8 + 1 = F_{16} + F_{13} + F_8 + F_5 + F_1$
- **Lekkerkerker's Theorem (1952).** The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.
- **Central Limit Type Theorem (KKMW 2010).** As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian.

Sequences Expressed in Terms of Bins

Another result by Zeckendorf states that if $\{a_n\}$ is a sequence of integers such that every positive integer can be written uniquely as a sum of nonadjacent terms in the sequence, then this sequence *must* be the Fibonacci sequence.
We can also view this construction as having a rule on what summands we can choose from bins of length 1: that no summands from adjacent bins may be chosen.

$$\underbrace{1}\,\underbrace{2}\,\underbrace{3}\,\underbrace{5}\cdots$$

Some natural questions to ask regarding these bin representation of sequences are:

- What happens when we allow the size of the bin to vary?
- What happens when the allowed numbers of summands per bin vary?
- In what situations do we retain uniqueness of decomposition?

Important Result

Let b_i be the number of terms in the i^{th} bin of a sequence, N the number of bins, and Y_i the number of summands chosen from the i^{th} bin. Then if $\sum_{i=1}^{\infty} \frac{1}{b_i}$ diverges, the distribution of the average number of summands in a decomposition converges to a Gaussian.

Choosing Arbitrarily Many Elements

We can generalize this notion further by choosing arbitrary numbers of elements from each bin. We let $A_i \subseteq \{0, 1, \ldots, b_i\}$ be a set of integers so that if $a \in A_i$, we may choose a summands from the i^{th} bin.

Important Result

Suppose $|A_n| \geq 2$. Then the distribution of the number of summands is Gaussian if the bin size b_n grows slower than $n^{\frac{1}{m_n - m_n'}}$.

Sketch of Proof:
The probability of choosing i elements from the n^{th} bin is $p(Y_n = i) = \frac{\binom{b_n}{i}}{\sum_{t \in A_n} \binom{b_n}{t}}$. We then find that

$$\sigma_n^2 = \mathbb{E}[Y_n^2] - \mathbb{E}[Y_n]^2 = \frac{\sum_{i,j \in A_n, i \neq j} (i-j)^2 \binom{b_n}{i}\binom{b_n}{j}}{2\left(\sum_{t \in A_n} \binom{b_n}{t}\right)^2}$$

$$\rho_n^{2+\delta} = \mathbb{E}\left[|Y_n - \mu_n|^{2+\delta}\right]$$

By asymptotically analyzing σ_n^2 and $\rho_n^{2+\delta}$ and applying the Lyapunov Central Limit Theorem, we find the above restriction for the growth of b_n.

Conclusion and Future Directions

A natural extension of these results is to examine the distribution of the average number of summands when we put adjacency conditions on the bins. However, since our variables $Y_1, \ldots, Y_n$ are dependent random variables, we must use a dependent version of the Central Limit Theorem:

Definition:
Let $\{X_i\}$ be a sequence of random variables. Then the ith α-mixing coefficient, α_i is defined to be $\alpha_i := \sup\{|\mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B)| : A \in X_{-\infty}^t, B \in X_{t+i}^{\infty}\}$, where X_a^b is the set of events involving finitely many random variables in the set $\{X_a, \ldots, X_b\}$.
We wish to bound α_i for constant bin sizes, essentially showing that the random variables are sufficiently independent when far enough apart.

References

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