Limiting Distributions in Generalized b-bin Zeckendorf Decompositions

Yujin Kim (yujin.kim@columbia.edu)
Eric Winsor (rcwnsr@umich.edu)

Collaborators: Granger Carty, Alexandre Gueganic, Alina Shubina, Shannon Sweitzer, and Jianing Yang

Faculty Mentor: Steven Miller

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Introduction

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Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8, 13...

Binning Perspective

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- **Generalization:** start with bins of specified size, and impose rules on how we choose summands from each bin. Obtain a sequence determined by these rules on the bins.

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...in which situations do we retain uniqueness of decomposition of the integers?
...in which situations do the distribution of the average number of summands in a decomposition converge to a Gaussian (CLT-type result)?

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- the size of the n^{th} bin is b_n
- the number of allowable elements we can choose from the n^{th} bin is $A_n \subset \{0, 1, 2, \ldots, b_n\}$
- we cannot take elements from two different bins unless there are at least $a \geq 0$ bins between them (adjacency condition)

Uniqueness

Theorem (Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)

A $(\{b_n\},\{A_n\},0)$-Sequence has uniqueness of decomposition (every number can be written and there is only one legal decomposition) if and only if for every positive n we have

$$A_n \in \{\{0,1\},\ \{0,1,\dots,b_n-1\},\ \{0,1,\dots,b_n\}\}.$$

In each of these cases for A_n, we derive a condition for the distribution of the number of summands whose largest summand is in bin N to converge to a Gaussian as $N \to \infty$.

Distributions of Summands: previous results

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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

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As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian.

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Note that this is equivalent to choosing summands from the first n Fibonacci numbers with the largest summand being F_n.

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- Distribution of summands after introducing binning?

Condition on Gaussianity, $A_n = \{0, 1\}$

Recall:

$$\textit{Uniqueness of decomposition}$$
$$\iff A_n \in \{\{0,1\},\ \{0,1,\dots,b_n-1\},\ \{0,1,\dots,b_n\}\}$$

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Recall:

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Theorem (Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)

Consider a $(\{b_n\},\{0,1\},0)$-Sequence. If $\sum_{n=1}^{\infty} 1/b_n$ diverges, then the distribution of the number of summands of integers whose largest summand is in bin N converges to a Gaussian as $N \to \infty$.

Condition on Gaussianity II, $A_n \in \{\{0, \dots, b_n - 1\}, \{0, \dots, b_n\}\}$

Theorem (Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)

Consider a $(\{b_n\}, \{A_n\}, 0)$-Sequence, where for all $n \in \mathbb{N}$, $b_n = n$, and $A_n \in \{\{0, \dots, n - 1\}, \{0, \dots, n\}\}$. The distribution of the number of summands of integers whose largest summand is in bin N converges to a Gaussian as $N \to \infty$.

Condition on Gaussianity III, A_n constant

Theorem (Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)

Assume A_n is a constant set A for all n. Then the distribution of the number of summands converges to Gaussian if $\sum \frac{1}{b_n^{m-m'}}$ diverges, where m is the maximal element of A and m' is the second maximal element.

Proof Method

We will need the following theorem for the proof:

Theorem (Lyapunov CLT)

Let $\{Y_1, Y_2, \dots\}$ be independent random variables, each with finite mean μ_i and variance σ_i^2. Define $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Then if there exists a $\delta > 0$ such that $\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^n \mathbb{E}(|Y_i - \mu_i|^{2+\delta}) = 0$, $\frac{1}{N} \sum_{i=1}^\infty Y_i$ converges to a Gaussian as $N \to \infty$.

Setting up the Random Variables

For an integer m let $Y_n(m) = 1$ if we use an element of the n^{th} bin in m's decomposition, and 0 otherwise; thus if the largest summand in m's decomposition is from bin N then the number of summands is $Y_1(m) + \cdots + Y_N(m)$

Proof Method (cont'd)

The probability of choosing i summands from the n-th bin is

$$p(Y_n = i) = \frac{\binom{b_n}{i}}{\sum_{t \in A_n} \binom{b_n}{t}},$$

Proof Method (cont'd)

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and the expectation values of Y_n and Y_n^2 are

$$\mathbb{E}[Y_n] = \frac{\sum_{t \in A_n} t\binom{b_n}{t}}{\sum_{t \in A_n} \binom{b_n}{t}}$$

$$\mathbb{E}[Y_n^2] = \frac{\sum_{t \in A_n} t^2\binom{b_n}{t}}{\sum_{t \in A_n} \binom{b_n}{t}}$$

Proof Method (cont'd)

We then find that

$$\sigma_n^2 = \mathbb{E}[Y_n^2] - \mathbb{E}[Y_n]^2$$
$$= \frac{\sum_{i,j \in A_n, i \neq j} (i-j)^2 \binom{b_n}{i}\binom{b_n}{j}}{2\left(\sum_{t \in A_n} \binom{b_n}{t}\right)^2},$$

and the absolute centered moment

$$\rho_n^{2+\delta} := \mathbb{E}\left[\left| Y_n - \mu_n \right|^{2+\delta} \right]$$
$$= \frac{\sum_{i \in A_n} \binom{b_n}{i} \left| \sum_{t \in A_n} (i-t)\binom{b_n}{t} \right|^{2+\delta}}{\left(\sum_{t \in A_n} \binom{b_n}{t}\right)^{3+\delta}}$$

Proof Method (cont'd)

We come to the conclusion of the theorem by analyzing σ_n^2 and $\rho_n^{2+\delta}$ asymptotically and applying the Lyapunov Central Limit Theorem.

Theorem (Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)

Consider a $(\{b_n\}, \{A_n\}, 0)$-Sequence, where for all $n \in \mathbb{N}$, $b_n = n$, and $A_n \in \{\{0, \ldots, n-1\}, \{0, \ldots, n\}\}$. The distribution of the number of summands of integers whose largest summand is in bin N converges to a Gaussian as $N \to \infty$.

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Future Work

We come to the conclusion of the theorem by analyzing σ_n^2 and $\rho_n^{2+\delta}$ asymptotically and applying the Lyapunov Central Limit Theorem.

Conjecture (Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)

Consider a $(\{b_n\}, \{A_n\}, 0)$-Sequence, where for all $k, n \in \mathbb{N}$, $k \le n$, $b_n = \lfloor n/k \rfloor$, and $A_n \in \{\{0, \ldots, n-1\}, \{0, \ldots, n\}\}$. The distribution of the number of summands of integers whose largest summand is in bin N converges to a Gaussian as $N \to \infty$. In fact, for any choice of $\delta > 0$, the Lyapunov condition is satisfied.

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