Limiting Distributions in Generalized $b$-bin Zeckendorf Decompositions

Yujin Kim (yujin.kim@columbia.edu)
Eric Winsor (rcwnsr@umich.edu)

Collaborators: Granger Carty, Alexandre Gueganic, Alina Shubina, Shannon Sweitzer, and Jianing Yang

Faculty Mentor: Steven Miller

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Introduction
Zeckendorf’s Theorem

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2017 = 1597 + 377 + 34 + 8 + 1 = F_{16} + F_{13} + F_{8} + F_{5} + F_{1}.
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Conversely, we can construct the Fibonacci sequence using this property:

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$$1, 2, 3, 5$$
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Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8, 13...
Binning Perspective

Example (Fibonacci):

1
2
3
5
···

Bins of constant length 1, construct $N$ by choosing summands from nonadjacent bins. (We take at most one summand from each bin)

Generalization: start with bins of specified size, and impose rules on how we choose summands from each bin. Obtain a sequence determined by these rules on the bins.
Example (Fibonacci):

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 5 & \ldots \\
2 & 3 & 5 & & & \\
3 & 5 & & & & \\
5 & & & & & \\
\end{array}
\]

Bins of constant length 1, construct $\mathbb{N}$ by choosing summands from *nonadjacent* bins.
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- **Example (Fibonacci):**

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Examples of Binning Generalization

Example: Bins of constant length $b_n = \frac{1}{2}$, may choose 0, 1 or 2 summands from each bin, summands from adjacent bins allowed

$1 \quad 2 \quad 4 \quad 8 \quad 16 \quad 32 \quad \ldots$

Example 2: Bins of length $b_n = \frac{n}{2}$, may choose 0, 1 or 2 summands from each bin, adjacent bins are not allowed

$1 \quad 2 \quad 3 \quad 4 \quad 6 \quad 8 \quad 16 \quad 20 \quad 24 \quad 28 \quad \ldots$
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```
1 2 4 8
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\begin{array}{ccc}
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1 2 3 4 6 8
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\[ \underline{1} \underline{2} \underline{3} \underline{4} \underline{6} \underline{8} \underline{16} 20 24 28 \ldots \]
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- the adjacency conditions on the bins (eg. summands may not be taken from adjacent bins)

. . . in which situations do we retain uniqueness of decomposition of the integers?
. . . in which situations do the distribution of the average number of summands in a decomposition converge to a Gaussian (CLT-type result)?
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A \((\{b_n\}, \{A_n\}, a)\)-Sequence has:
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A \( \{b_n\}, \{A_n\}, a \)-Sequence has:

- the size of the \( n^{\text{th}} \) bin is \( b_n \)
- the number of allowable elements we can choose from the \( n^{\text{th}} \) bin is \( A_n \subset \{0, 1, 2, \ldots, b_n\} \)
- we cannot take elements from two different bins unless there are at least \( a \geq 0 \) bins between them (adjacency condition)
Uniqueness

**Theorem (Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)**

A \((\{b_n\}, \{A_n\}, 0)\)-Sequence has uniqueness of decomposition (every number can be written and there is only one legal decomposition) if and only if for every positive \(n\) we have

\[ A_n \in \\{\{0, 1\}, \{0, 1, \ldots, b_n - 1\}, \{0, 1, \ldots, b_n\}\} . \]

In each of these cases for \(A_n\), we derive a condition for the distribution of the number of summands whose largest summand is in bin \(N\) to converge to a Gaussian as \(N \to \infty\).
Distributions of Summands: previous results

Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in \([F_n, F_{n+1})\) tends to \(\frac{n}{\phi^2 + 1} \approx 2.276\), where \(\phi = \frac{1 + \sqrt{5}}{2}\) is the golden mean.

Central Limit Type Theorem (KKMW 2010)

As \(n \to \infty\), the distribution of the number of summands in the Zeckendorf decomposition for integers in \([F_n, F_{n+1})\) is Gaussian.

Remark:

Note that this is equivalent to choosing summands from the first \(n\) Fibonacci numbers with the largest summand being \(F_n\).
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Distribution of summands after introducing binning?
Condition on Gaussianity, $A_n = \{0, 1\}$

Recall:

Uniqueness of decomposition

$\iff A_n \in \{\{0, 1\}, \{0, 1, \ldots, b_n - 1\}, \{0, 1, \ldots, b_n\}\}$
Condition on Gaussianity, $A_n = \{0, 1\}$

Recall:

Uniqueness of decomposition

$$A_n \in \{\{0, 1\}, \{0, 1, \ldots, b_n - 1\}, \{0, 1, \ldots, b_n\}\}$$

Theorem (Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)

Consider a ($\{b_n\}, \{0, 1\}, 0$)-Sequence. If $\sum_{n=1}^{\infty} 1/b_n$ diverges, then the distribution of the number of summands of integers whose largest summand is in bin $N$ converges to a Gaussian as $N \to \infty$. 
Condition on Gaussianity II, $A_n \in \{\{0, \ldots, b_n - 1\}, \{0, \ldots, b_n\}\}$

Theorem (Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)

Consider a $({b_n}, {A_n}, 0)$-Sequence, where for all $n \in \mathbb{N}$, $b_n = n$, and $A_n \in \{\{0, \ldots, n - 1\}, \{0, \ldots, n\}\}$. The distribution of the number of summands of integers whose largest summand is in bin $N$ converges to a Gaussian as $N \to \infty$. 
Condition on Gaussianity III, $A_n$ constant

**Theorem (Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)**

Assume $A_n$ is a constant set $A$ for all $n$. Then the distribution of the number of summands converges to Gaussian if

$$\sum \frac{1}{b_n^{m-m'}}$$

diverges, where $m$ is the maximal element of $A$ and $m'$ is the second maximal element.
Proof Method

We will need the following theorem for the proof:

**Theorem (Lyapunov CLT)**

Let \( \{Y_1, Y_2, \ldots\} \) be independent random variables, each with finite mean \( \mu_i \) and variance \( \sigma_i^2 \). Define \( s_n^2 = \sum_{i=1}^{n} \sigma_i^2 \). Then if there exists a \( \delta > 0 \) such that

\[
\lim_{n \to \infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} \mathbb{E}(|Y_i - \mu_i|^{2+\delta}) = 0, \quad \frac{1}{N} \sum_{i=1}^{\infty} Y_i \text{ converges to a Gaussian as } N \to \infty.
\]
Setting up the Random Variables

For an integer $m$ let $Y_n(m) = 1$ if we use an element of the $n^{th}$ bin in $m$’s decomposition, and 0 otherwise; thus if the largest summand in $m$’s decomposition is from bin $N$ then the number of summands is $Y_1(m) + \cdots + Y_N(m)$.
The probability of choosing \( i \) summands from the \( n \)-th bin is

\[
p( Y_n = i ) = \frac{ \binom{b_n}{i} }{ \sum_{t \in A_n} \binom{b_n}{t} },
\]
The probability of choosing \( i \) summands from the \( n \)-th bin is

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\]

and the expectation values of \( Y_n \) and \( Y_n^2 \) are

\[
\mathbb{E}[Y_n] = \frac{\sum_{t \in A_n} t \binom{b_n}{t}}{\sum_{t \in A_n} \binom{b_n}{t}} \quad \mathbb{E}[Y_n^2] = \frac{\sum_{t \in A_n} t^2 \binom{b_n}{t}}{\sum_{t \in A_n} \binom{b_n}{t}}
\]
We then find that
\[
\sigma_n^2 = \mathbb{E}[Y_n^2] - \mathbb{E}[Y_n]^2
\]
\[
= \sum_{i,j \in A_n, i \neq j} (i - j)^2 (b_n_i)(b_n_j) \cdot 
\frac{2 \left( \sum_{t \in A_n} (b_n_t) \right)^2}{2 \left( \sum_{t \in A_n} (b_n_t) \right)^2},
\]
and the absolute centered moment
\[
\rho_n^{2+\delta} := \mathbb{E} \left[ |Y_n - \mu_n|^{2+\delta} \right]
\]
\[
= \sum_{i \in A_n} \left( \frac{b_n_i}{b_n} \right) \left| \sum_{t \in A_n} (i - t) (b_n_t) \right|^{2+\delta}
\]
\[
= \frac{\left( \sum_{t \in A_n} (b_n_t) \right)^{3+\delta}}{\left( \sum_{t \in A_n} (b_n_t) \right)}^{3+\delta}
\]
Proof Method (cont’d)

We come to the conclusion of the theorem by analyzing $\sigma_n^2$ and $\rho_n^{2+\delta}$ asymptotically and applying the Lyapunov Central Limit Theorem.

**Theorem (Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)**

Consider a $\left(\{b_n\}, \{A_n\}, 0\right)$-Sequence, where for all $n \in \mathbb{N}$, $b_n = n$, and $A_n \in \{\{0, \ldots, n-1\}, \{0, \ldots, n\}\}$. The distribution of the number of summands of integers whose largest summand is in bin $N$ converges to a Gaussian as $N \to \infty$. 
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We come to the conclusion of the theorem by analyzing $\sigma_n^2$ and $\rho_n^{2+\delta}$ asymptotically and applying the Lyapunov Central Limit Theorem.

**Conjecture**
*(Carty-Gueganic-K-M-Shubina-Sweitzer-W-Yang)*

Consider a $\langle \{b_n\}, \{A_n\}, 0 \rangle$-Sequence, where for all $k, n \in \mathbb{N}$, $k \leq n$, $b_n = \lfloor n/k \rfloor$, and $A_n \in \{ \{0, \ldots, n-1\}, \{0, \ldots, n\} \}$. The distribution of the number of summands of integers whose largest summand is in bin $N$ converges to a Gaussian as $N \to \infty$. In fact, for any choice of $\delta > 0$, the Lyapunov condition is satisfied.
References

- P. Billingsley, *Probability and Measure* (1979), pages 377-381


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Thank You!