On Summand Minimality of Generalized Zeckendorf Decompositions

Katherine Cordwell\textsuperscript{1}, Magda Hlavacek\textsuperscript{2}
SMALL REU, Williams College

ktcordwell@gmail.com
mhlavacek@hmc.edu

\textsuperscript{1}University of Maryland, College Park
\textsuperscript{2}Harvey Mudd College

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Fibonacci Sequence

\[ F_{n+1} = F_n + F_{n-1} \]

1, 2, 3, 5, 8, 13, 21, ...
Zeckendorf Decompositions

Theorem (Zeckendorf)

Every positive integer has a unique representation as a sum of nonadjacent Fibonacci numbers.

Edouard Zeckendorf
Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, nonlegal decomposition, three summands
Definition

The Zeckendorf decomposition is summand minimal, because the Zeckendorf decomposition of any positive integer $n$ uses the fewest summands out of any decomposition of $n$ into Fibonacci numbers.
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Overall Question
What other recurrences are summand minimal?
Positive Linear Recurrence Sequences

Definition

A positive linear recurrence sequence (PLRS) is the sequence given by a recurrence $a$ of the following form:

$$a_n = c_1 a_{n-1} + \cdots + c_t a_{n-t}$$

where all $c_i \geq 0$ and $c_1, c_t > 0$. 
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where all \( c_i \geq 0 \) and \( c_1, c_t > 0 \). We use **ideal initial conditions** \( a_{-(n-1)} = 0, \ldots, a_{-1} = 0, a_0 = 1 \).
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where all $c_i \geq 0$ and $c_1, c_t > 0$. We use ideal initial conditions $a_{-(n-1)} = 0, \ldots, a_{-1} = 0, a_0 = 1$.

Definition

We call $(c_1, \ldots, c_t)$ the signature of the sequence.
Definition

For some sequence \( \{a_i\} \), suppose we have:

\[
    n = b_k a_k + b_{k-1} a_{k-1} + \cdots + b_0 a_0
\]

Then we call \([b_k, b_{k-1}, \ldots, b_0, \infty]\) a representation of \( n \) over \( \{a_0\} \).
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Then we call \([b_k, b_{k-1}, \ldots, b_0, \infty]\) a representation of \(n\) over \(\{a_0\}\).

Example

18 = \(F_6 + F_4\) and \(F_6 + F_3 + F_2\)

We denote these two representations of 18 by:

\([1, 0, 1, 0, 0, 0, 0, \infty]\) and \([1, 0, 0, 1, 1, 0, 0, \infty]\)
Allowable Blocks

Definition

Given a PLRS with signature \((c_1, \ldots, c_t)\), we say that \([b_1, \ldots, b_k]\) is an **allowable block** if \(k \leq t\) and \(b_i = c_i\) for \(i < k\) and \(0 \leq b_k < c_k\).
Allowable Blocks

**Definition**

Given a PLRS with signature \((c_1, \ldots, c_t)\), we say that \([b_1, \ldots, b_k]\) is an **allowable block** if \(k \leq t\) and \(b_i = c_i\) for \(i < k\) and \(0 \leq b_k < c_k\).

**Example**

Signature: \((1, 2, 1)\)
Legal blocks: \([0], [1, 0], [1, 1], [1, 2, 0]\)
Generalized Zeckendorf Decompositions

Definition

Given a positive linear recurrence, a representation of a positive integer $n$ is a **generalized Zeckendorf decomposition** (GZD) if it can be formed as a tiling of a set of allowable blocks.
Generalized Zeckendorf Decompositions

Definition
Given a positive linear recurrence, a representation of a positive integer \( n \) is a **generalized Zeckendorf decomposition** (GZD) if it can be formed as a tiling of a set of allowable blocks.

Theorem (Miller et. al., Hamlin)
*Given any positive linear recurrence, each positive integer \( n \) has a unique generalized Zeckendorf decomposition.*
Not All Recurrences are Summand Minimal

Recurrence: \( f_n = f_{n-1} + 2f_{n-2} + f_{n-3} \)

Signature: (1, 2, 1)
Sequence: 1, 1, 3, 6, 13, ...
Allowable blocks: [0], [1, 0], [1, 1], [1, 2, 0]
Not All Recurrences are Summand Minimal

Recurrence: \( f_n = f_{n-1} + 2f_{n-2} + f_{n-3} \)

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GZD
- \( 6 + 2(3) \)
- [1, 2, 0, 0]
- Three summands
Not All Recurrences are Summand Minimal

Recurrence: $f_n = f_{n-1} + 2f_{n-2} + f_{n-3}$

Signature: (1, 2, 1)
Sequence: 1, 1, 3, 6, 13, ...
Allowable blocks: [0], [1, 0], [1, 1], [1, 2, 0]
Main Result

Theorem

Suppose we have a PLRS with signature \((c_1, c_2, \ldots, c_t)\)\n
The corresponding GZD of each positive integer \(n\) is summand minimal if and only if:

\[
c_1 \geq c_2 \geq \cdots \geq c_t.
\]
Borrow and Carry

Signature: \((c_1, c_2, \ldots, c_t)\)

\[ [1, 0, \ldots, 0] \quad \text{borrow} \quad [0, c_1, \ldots, c_t] \]

\[ [0, c_1, \ldots, c_t] \quad \text{carry} \quad [1, 0, \ldots, 0] \]
Proof Sketch

**Theorem (\(\iff\))**

*If the signature of a recurrence is weakly decreasing, then the recurrence is summand minimal.*
Theorem (\(\Rightarrow\))

*If the signature of a recurrence is weakly decreasing, then the recurrence is summand minimal.*

**Proof.**

Start with any representation and use successive borrows and carries to reach the GZD. Because after borrowing we can always carry, we never increase the number of summands.
**Sufficiency Example**

Signature: (3, 2, 1)
Sequence: 1, 3, 11, 40, ...
Start with: 15

\[
\begin{align*}
[5, 0, \infty] & \quad 15 = 5 \cdot 3
\end{align*}
\]
Sufficiency Example

Signature: (3, 2, 1)
Sequence: 1, 3, 11, 40, ...
Start with: 15

\[ [5, 0, \infty] \quad \text{borrow} \quad [4, 3, \infty] \]

\[ 15 = 5 \cdot 3 \]
\[ 15 = 4 \cdot 3 + 3 \cdot 1 \]
Sufficiency Example

Signature: (3, 2, 1)
Sequence: 1, 3, 11, 40, ...
Start with: 15

\[ [5, 0, \infty] \]

borrow

\[ [4, 3, \infty] \]

15 = 5 \times 3

\[ [1, 1, 1, \infty] \]

carry

15 = 4 \times 3 + 3 \times 1

15 = 1 \times 11 + 1 \times 3 + 1 \times 1
Theorem ($\iff$)

*If a recurrence is summand minimal, then its signature is weakly decreasing.*
Theorem ($\iff$)

If a recurrence is summand minimal, then its signature is weakly decreasing.

Proof.

Two cases: $c_1 > 1$ or $c_1 = 1$. 
Theorem (⇐)

If a recurrence is summand minimal, then its signature is weakly decreasing.

Proof.

Two cases: $c_1 > 1$ or $c_1 = 1$.
If $c_1 > 1$, then the general idea is to construct a non-legal representation and show that the corresponding legal representation uses more summands.
Theorem $(\iff)$

If a recurrence is summand minimal, then its signature is weakly decreasing.

Proof.

Two cases: $c_1 > 1$ or $c_1 = 1$.
If $c_1 > 1$, then the general idea is to construct a non-legal representation and show that the corresponding legal representation uses more summands.
If $c_1 = 1$, then we use a growth rate argument to demonstrate the existence of a non-legal representation.
Case 1: $c_1 > 1$

Further subcases. One example:

Signature: $(2, 1, 3)$
Sequence: $0, 0, 1, 2, 5, 15, 41, 112, ...$
Blocks: $[0], [1], [2, 0], [2, 1, 0], [2, 1, 1], [2, 1, 2]$
Case 2: \( c_1 = 1 \)

Assume \( c_1 = 1 \). What are good subcases? Let \( k > 1 \).

- \( \{1, k, \ldots\} \)
- \( \{1, \ldots, 1, k, \ldots\}, \ a \geq 2 \)
- \( \{1, \ldots, 1, 0, \ldots, 0, \ldots, 1, \ldots, 1, 0, \ldots, 0, k, \ldots\} \) \( a_1 > 1 \)
- \( \{1, 0, \ldots, 0, k, \ldots\} \)
Example Construction

Signature: \((1, ..., 1, 0, ..., 0, ..., 1, ..., 1, k, ...), a_1 > 1\)

Non-legal representation:

\([1, ..., 1, 0, ..., 0, ..., 1, ..., 1, 2, 0, ..., 0, \infty]\)

Change in summands: \(a_1 - 1\)
Subcases Start to get Overwhelming

\begin{itemize}
  \item $c_0 = 1$?
  \item $\exists c_i > 1$?
  \item $c_2 > 1$?
  \item $\exists 0$ before $c_i > 1$
  \item $c_2 = 1$
  \item $11\ldots 00..01..1k$
  \item $1100..01..10..0k$
\end{itemize}

$1k \rightarrow \text{Proven}$

$11..1k \rightarrow \text{Proven}$
Definition

The characteristic polynomial of a recurrence with signature \((c_1, c_2, \ldots, c_k)\) is

\[ x^k - c_1 x^{k-1} - \cdots - c_k. \]
Solution: Growth Rates!

**Definition**

The characteristic polynomial of a recurrence with signature $(c_1, c_2, \ldots, c_k)$ is

$$x^k - c_1 x^{k-1} - \cdots - c_k.$$

**Theorem**

*Given a PLRS with a signature of the form $(1, c_2, \ldots)$, the characteristic polynomial has a unique largest positive root $\alpha > 1$. For large $n$,*

$$a_n \approx C\alpha^n.$$
Counterexample: Trying to represent $2a_n$

**Theorem**

For every non-weakly-decreasing signature with $c_1 = 1$, then there exists some $n$ for which the GZD of $2a_n$ has at least 3 summands.
Counterexample: Trying to represent \(2a_n\)

**Theorem**

*For every non-weakly-decreasing signature with \(c_1 = 1\), then there exists some \(n\) for which the GZD of \(2a_n\) has at least 3 summands.*

If summand minimal, the GZD of \(2a_n\) must have only 1 or 2 summands:

- \([1, 0, 0, \ldots]\)
- \([1, 0, \ldots, 1, 0, \ldots]\)
### Representations of $2a_n$

Growth rate arguments give three specific forms:

- $2a_n = a_{n+r}$
- $2a_n = a_{n+r} + a_{n-s}$
- $2a_n = a_{n+r} + a_{n-s+1}$

for some fixed $r$, $s$. 
Representations of $2a_n$

Growth rate arguments give three specific forms:

- $2a_n = a_{n+r}$
- $2a_n = a_{n+r} + a_{n-s}$
- $2a_n = a_{n+r} + a_{n-s+1}$

for some fixed $r, s$.

These recurrences have different growth rates; only one can correspond to our sequence.

For all $n > N$, every representation of $2a_n$ must be of the same form.
The characteristic polynomial of a truncated sequence must divide exactly one of the following characteristic polynomials:

- \( x^r - 2 \)
- \( x^r - 2x^s - 1 \)
- \( x^r - 2x^{s-1} - 1 \)

By a result of Schinzel on the factorization of these polynomials, this cannot be the case.
The characteristic polynomial of a truncated sequence must divide exactly one of the following characteristic polynomials:

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Selected References


Sufficiency Example

Signature: (3, 2, 1)
Sequence: 1, 3, 11, 40, 145 ...
Start with: 160

[4, 0, 0, 0, ∞]  
160 = 4•40
Sufficiency Example

Signature: (3, 2, 1)
Sequence: 1, 3, 11, 40, 145 ...
Start with: 160

\[ [4, 0, 0, 0, \infty] \]

160 = 4 \cdot 40

\[ [3, 3, 2, 1, \infty] \]

borrow
Sufficiency Example

Signature: (3, 2, 1)
Sequence: 1, 3, 11, 40, 145 ...
Start with: 160

\[
\begin{align*}
[4, 0, 0, 0, \infty] & \quad \text{borrow} \\
[3, 3, 2, 1, \infty] & \quad \text{carry} \\
[1, 0, 1, 1, 1, \infty] & \\
\end{align*}
\]

160 = 4\cdot40

Check: 1 + 3 + 11 + 145 = 160
Case 1: $c_1 > 1$ Further example

Further subcases. One example:

Signature: $(c_1, c_2, \ldots, c_t)$, some $c_i > c_1$
Non-legal representation: $[0, c_1, \ldots, c_{i-2}, c_{i-1} + 1, 0, \infty]$  

$$[0, c_1, \ldots, c_{i-2}, c_{i-1} + 1, 0, \infty]$$

\[ \downarrow \text{borrow} \]

$$[0, c_1, \ldots, c_{i-2}, c_{i-1}, c_1, \infty]$$

Net change in summands: $c_1 - 1 \geq 1$