Notes on eigenvalue distributions for the classical compact groups

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September 3, 2008

Contents

1 Some notation

Before we get started we record some notation that will be used in these notes. This section is merely to serve as a convenient reference; the notions are defined at the appropriate places in the notes.

- The sine ratios are:

$$S(x) = \frac{\sin \pi x}{\pi x}$$

$$S_N(x) = \frac{\sin Nx/2}{\sin x/2}$$

- $G(N)$ stands for one of the groups $U(N)$, $USp(2N)$, $SO(2N)$, $SO(2N+1)$ and G by itself stands for one of the symmetry types U (unitary), Sp (symplectic), O (orthogonal) even, O (orthogonal) odd
- The kernel functions are

$$K_{U(N)}(x,y) = S_N(y-x)$$

$$K_{SO(2N)}(x,y) = \frac{S_{2N-1}(y-x) + S_{2N-1}(y+x)}{2}$$

$$K_{USp(2N)}(x,y) = \frac{S_{2N+1}(y-x) - S_{2N+1}(y+x)}{2}$$

$$K_{SO(2N+1)}(x,y) = \frac{S_{2N}(y-x) - S_{2N}(y+x)}{2}.$$

- The scaled limit of these kernel functions are

$$K_U(x,y) = S(y-x)$$

$$K_{Sp}(x,y) = S(y-x) - S(y+x)$$

$$K_{O,even}(x,y) = S(y-x) + S(y+x)$$

$$K_{O,odd}(x,y) = S(y-x) - S(y+x)$$

- For an interval J, the integral operator $K_{J,G(N)}$ is defined by

$$(K_{J,G(N)}f)(x) \quad = \quad \int_J K_{G(N)}(x,y) f(y)\ dy$$

for functions f integrable on J, and similarly the operator $K_{J,G}$ is defined by

$$(K_{J,G}f)(x) \quad = \quad \int_J K_G(x,y) f(y)\ dy$$

These operators have eigenvalues denoted by $\lambda_{j,G(N)}(J)$ $(j = 1,2,\dots,N)$ and $\lambda_{j,G}(J)$, $(j = 1,2,3,\dots)$ respectively.

- The Chebyshev polynomials are $T_n(x), U_n(x)$, and $V_n(x)$ where

$$T_n(\cos\theta) = \cos n\theta$$

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

$$V_n(\cos\theta) = U_{2n}\big(\cos\frac{\theta}{2}\big) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}.$$

- We let $\mu_{G(N),j}(s)$ be the density function for the jth nearest neighbor spacing for eigenangles of $G(N)$ and $\mu_{G,j}(s)$ is the large–N–scaled–limit of this density function. Similarly, $\nu_{G(N),j}(s)$ is the density of the jth lowest eigenangle for $G(N)$ and $\nu_{G,j}(s)$ is its scaled limit.

- We let $E_{G(N)}(J,n)$ be the measure of the set of matrices $X \in G(N)$ which have precisely n eigenangles in the set J.

2 Introduction

In 1972 the fortuitous introduction of Montgomery and Dyson served also as an introduction of the worlds of analytic number theory and random matrix theory. The symbiosis between these two subjects developed slowly for the next 25 years with the principal developments being the numerical work of Odlyzko and the calculations of the third and higher correlations of the Riemann zeta-function (and other L-functions) by Hejhal, and Rudnick-Sarnak.

Around 1998, there were two very important developments that have stimulated a great deal of subsequent work. One was the theory of symmetry types associated to families of L-functions by Katz and Sarnak. The other was the relationship between moments of characteristic polynomials and moments of the Riemann zeta-function and of families of L-functions found by Keating and Snaith.

While we still do not understand why there is such a strong connnection between random matrix theory and families of L-functions, we do realize that random matrix theory provides models for a wide range of statistical behavior of these families. Consequently, we can now confidently predict the answer to any number of difficult questions about L-functions which 10 years ago seemed hopelessly impossible.

The purpose of these notes is to provide an introduction to the random matrix aspects of the book [KaSa] by Katz and Sarnak on symmetry types associated with families of L-functions. In particular, we will develop here some of the basic tools needed to understand the beginnings of computing statistics of eigenvalues of unitary, orthogonal, and symplectic groups of matrices. The four statistics we are interested in computing are n-correlation, n–level density, jth nearest neighbor, and jth lowest eigenvalue.

The main goals of these notes are (a) to show how to rewrite the basic Weyl integration formula for each of our groups $G(N)$ as a determinant of a "kernel" function $K_{G(N)}$ (derived in sections 2 – 5, equations (9), (17), (18), and (19)); (b) to use Gaudin's lemma to compute level densities and correlations (derived in sections 6 – 8, equations (28), (29), and (30)); (c) to use the combinatorial identity (34) to deduce the mth nearest neighbor statistic from the correlations (derived in section 9.1, equation (35)); and (d) to use Gram's identity to write the neighbor and lowest eigenvalue statistics in terms of derivatives of infinite products of eigenvalues of simple operators (derived in sections 9.2 – 9.5, equations (47) and (50)).

3 Definitions and Haar measures

3.1 Unitary

If X is an $N \times N$ matrix with complex entries $X = (x_{jk})$, we let X^* be its conjugate transpose, i.e. $X^* = (x^*_{jk})$ where $x^*_{jk} = \overline{x_{kj}}$. X is said to be unitary if $XX^* = I$. We let $U(N)$ denote the group of all $N \times N$ unitary matrices. This is a compact Lie group and has a Haar measure which allows us to do analysis.

All of the eigenvalues of $X \in U(N)$ have absolute value 1; we write them as

$$e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N}$$

with

$$0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_N < 2\pi. \tag{1}$$

The eigenvalues of X^* are $e^{-i\theta_1}, \dots, e^{-i\theta_N}$. Clearly, the determinant, $\det X = \prod_{n=1}^N e^{i\theta_n}$ of a unitary matrix is a complex number with absolute value equal to 1.

For any sequence of N points on the unit circle there are matrices in $U(N)$ with these points as eigenvalues. The collection of all matrices with the same set of eigenvalues constitutes a conjugacy class in $U(N)$. Thus, the set of conjugacy classes can be identified with the collection of sequences of N points on the unit circle.

We are interested in computing various statistics about these eigenvalues. Consequently, we identify all matrices in $U(N)$ that have the same set of eigenvalues. Weyl's integration formula gives a simple way to perform averages over $U(N)$ for functions f that are constant on conjugacy classes. Such functions are called 'class functions'. Note that f being constant on conjugacy classes entails that $f(\theta_1, \ldots, \theta_N)$ is necessarily symmetric in its N variables. Weyl's formula asserts that for such an f,

$$\int_{U(N)} f(X)\, dX = \int_{[0,2\pi]^N} f(\theta_1, \ldots, \theta_N) \prod_{1 \le j < k \le N} \left| e^{i\theta_k} - e^{i\theta_j} \right|^2 \frac{d\theta_1 \ldots d\theta_N}{N!(2\pi)^N}.$$

Notice that we have used X to represent a variable element of $U(N)$ and dX to denote the Haar measure. If we want to emphasize the group $U(N)$ we will designate the Haar measure by $dX_{U(N)}$.

3.2 Orthogonal and Symplectic

A unitary matrix X is said to be *orthogonal* if $XX^t = I$, where X^t denotes the transpose of X. Orthogonality for a unitary matrix implies that $X^t = X^*$ or $\overline{X} = X$. In other words any real unitary matrix is orthogonal. We let $SO(N)$ denote the subgroup of $U(N)$ consisting of $N \times N$ orthogonal matrices with determinant 1.

We want to distinguish these two cases. Thus, we consider $SO(2N)$ (even orthogonal) and $SO(2N+1)$ (odd orthogonal).

For any complex eigenvalue of an orthogonal matrix, its complex conjugate is also an eigenvalue. The eigenvalues of $X \in SO(2N)$ can be written as

$$e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$$

with

$$0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_N \le \pi.$$

The Weyl integration formula for integrating a symmetric function $f(X) = f(\theta_1, \ldots, \theta_N)$ over $SO(2N)$ is

$$\int_{SO(2N)} f(X)\ dX = \frac{2^{(N-1)^2}}{\pi^N N!} \int_{[0,\pi]^N} f(\theta_1, \ldots, \theta_N) \times \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 d\theta_1 \ldots d\theta_N.$$

The eigenvalues of $X \in SO(2N+1)$ can be written as

$$1, e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$$

with

$$0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_N \le \pi.$$

The Weyl integration formula for integrating a symmetric function $f(X) = f(\theta_1, \ldots, \theta_N)$ over the space $SO(2N+1)$ is

$$\int_{SO(2N+1)} f(X)\ dX = \frac{2^{N^2}}{\pi^N N!} \int_{[0,\pi]^N} f(\theta_1, \ldots, \theta_N) \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 \times \prod_{h=1}^{N} \sin^2\frac{\theta_h}{2}\ d\theta_1 \ldots d\theta_N.$$

A unitary matrix X is said to be *symplectic* if $XZX^t = Z$ where

$$Z = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}.$$

A symplectic matrix necessarily has determinant equal to 1. The *symplectic group* $\mathrm{USp}(2N)$ is the subgroup of $2N \times 2N$ symplectic matrices. The eigenvalues of a symplectic matrix are

$$e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$$

with

$$0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_N \le \pi.$$

The Weyl integration formula for integrating a symmetric function $f(X) = f(\theta_1, \ldots, \theta_N)$ over $USp(2N)$ is

$$\int_{USp(2N)} f(X)\, dX = \frac{2^{N^2}}{\pi^N N!} \int_{[0,\pi]^N} f(\theta_1, \ldots, \theta_N) \prod_{1 \le j < k \le N} (\cos\theta_k - \cos\theta_j)^2 \prod_{h=1}^{N} \sin^2\theta_h \, d\theta_1 \ldots d\theta_N.$$

4 Vandermonde determinants and orthogonal polynomials

We occasionally use the notation $(f(j,k))_{j,k}$ to denote the matrix whose j,k entry is $f(j,k)$.

We recall the basic fact about Vandermonde determinants. For any set N-tuple of complex numbers $(x_1, \ldots, x_N)$ let

$$\Delta(x_1, \ldots, x_N) = \det_{N \times N} \left(x_k^{j-1}\right)_{jk}. \tag{2}$$

Then

$$\Delta(x_1, \ldots, x_N) = \prod_{1 \le j < k \le N} (x_k - x_j). \tag{3}$$

To prove this, one observes that both sides are homogeneous polynomials of total degree $N(N-1)/2$ which vanish whenever $x_j = x_k$. This fact identifies the two sides up to a constant factor. That the coefficient of $x_N^{N-1} x_{N-1}^{N-2} \ldots x_2$ is 1 in both expressions completes the proof.

Observe that

$$\prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^2 = |\Delta(e^{i\theta_1}, \ldots, e^{i\theta_N})|^2 \tag{4}$$

and

$$\prod_{1 \le j < k \le N} (\cos\theta_k - \cos\theta_j)^2 = \Delta(\cos\theta_1, \ldots, \cos\theta_N)^2. \tag{5}$$

Useful in our calculations will be the

Lemma 1 (Transposing Lemma) *We have*

$$\det_{N\times N}(\phi_{j-1}(x_k))\det_{N\times N}(\psi_{j-1}(x_k)) = \det_{N\times N}\Big(\sum_{n=1}^{N}\phi_{n-1}(x_j)\psi_{n-1}(x_k)\Big). \tag{6}$$

This identity just follows by using the fact that the determinant of a matrix and its transpose are the same, and matrix multiplication. Specifically,

$$\begin{aligned}\det_{N\times N}(\phi_{j-1}(x_k))\det_{N\times N}(\psi_{j-1}(x_k)) &= \det_{N\times N}(\phi_{n-1}(x_j))_{j,n}\det_{N\times N}(\psi_{n-1}(x_k))_{n,k}\\ &= \det\Big(\sum_{n=1}^{N}\phi_{n-1}(x_j)\psi_{n-1}(x_k)\Big)_{j,k}.\end{aligned}$$

4.1 An alternate formula for the Haar measure on $U(N)$

In order to compute the statistics we desire, we require an alternate formula for the Haar measure. The Transposing Lemma implies the identity

$$\prod_{1\le j<k\le N}|e^{i\theta_k}-e^{i\theta_j}|^2 = \det_{N\times N}\big(S_N(\theta_k-\theta_j)\big) \tag{7}$$

where

$$S_N(\theta) = \frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}. \tag{8}$$

From this identity we have

$$dX_{U_N} = \frac{d\theta_1\ldots d\theta_N}{(2\pi)^N N!}\det_{N\times N}\big(S_N(\theta_k-\theta_j)\big). \tag{9}$$

To prove this we apply the Transposing Lemma with $\phi_j(x_k) = e^{ij\theta_k}$ and $\psi_j(x_k) = e^{-ij\theta_k}$ and use the fact that

$$\sum_{n=1}^{N} e^{i(n-1)\theta} = \frac{e^{iN\theta}-1}{e^{i\theta}-1} = \frac{e^{iN\theta/2}}{e^{i\theta/2}}\frac{e^{iN\theta/2}-e^{-iN\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}} = e^{i(N-1)\theta/2}S_N(\theta)$$

from which

$$\begin{aligned}|\det(e^{i(j-1)\theta_k})|^2 &= \det\Big(\sum_{n=1}^{N}e^{i(n-1)(\theta_j-\theta_k)}\Big)_{j,k}\\ &= \det\big(e^{iN(\theta_j-\theta_k)/2}S_N(\theta_j-\theta_k)\big)\\ &= \det\big(S_N(\theta_j-\theta_k)\big);\end{aligned}$$

the last line holds by factoring out $e^{iN\theta_j/2}$ from the jth row and $e^{-iN\theta_k/2}$ from the kth column and observing that the product of all of these factors is 1.

For future reference we introduce the notation

$$S(x) = \frac{\sin \pi x}{\pi x}. \tag{10}$$

4.2 Alternate formulas for orthogonal and symplectic Haar measures

Now we give alternate formulas for our other measures. To accomplish this, it is helpful to first recall the basic properties of the Tchebychev polynomials. Let $T_n(x)$ be the (Chebyshev) polynomial of degree n for which

$$T_n(\cos\theta) = \cos n\theta \tag{11}$$

and $U_n(x)$ is the polynomial of degree n for which

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{12}$$

Thus, $T_0(x) = 1, T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$ and so on and $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, $U_3(x) = 8x^3 - 4x$, and so on. From $\cos(n+1)\theta = \cos n\theta \cos\theta - \sin n\theta \sin\theta$ and $\sin(n+1)\theta = \sin n\theta \cos\theta + \cos n\theta \sin\theta$ it is easy to see that

$$T_{n+1}(x) = xT_n(x) - (1-x^2)U_{n-1}(x)$$

and

$$U_n(x) = xU_{n-1}(x) + T_n(x).$$

Thus,

$$\begin{aligned} T_{n+2}(x) &= xT_{n+1}(x) - (1-x^2)U_n(x) \\ &= xT_{n+1}(x) - (1-x^2)(xU_{n-1}(x) + T_n(x)) \\ &= xT_{n+1}(x) - (1-x^2)T_n(x) + x(T_{n+1}(x) - xT_n(x)) \\ &= 2xT_{n+1}(x) - T_n(x). \end{aligned}$$

Similarly, $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$. Notice that the leading coefficient in $T_n(x)$ is 2^{n-1} and in $U_n(x)$ it is 2^n.

Finally, we let $V_n(x)$ be the polynomial of degree n for which

$$V_n(\cos\theta) = U_{2n}\big(\cos\frac{\theta}{2}\big) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}. \tag{13}$$

It can be shown that $V_n(x) = 2^n x^n + \ldots$ has leading coefficient 2^n.

Now $\prod_{1\le j<k\le N}(\cos\theta_k - \cos\theta_j) = \Delta(\cos\theta_1, \ldots, \cos\theta_N)$. Let $x_j = \cos\theta_j$ for convenience. Then, by elementary row operations, $\Delta(x_1, \ldots, x_N)$

$$\begin{aligned}
&= \det\begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_N \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{N-2} & x_2^{N-2} & \ldots & x_N^{N-2} \\ x_1^{N-1} & x_2^{N-1} & \ldots & x_N^{N-1} \end{pmatrix} \\
&= \frac{1}{2^{N-2}}\det\begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_N \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{N-2} & x_2^{N-2} & \ldots & x_N^{N-2} \\ 2^{N-2}x_1^{N-1} & 2^{N-2}x_2^{N-1} & \ldots & 2^{N-2}x_N^{N-1} \end{pmatrix} \\
&= \frac{1}{2^{N-2}}\det\begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_N \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{N-2} & x_2^{N-2} & \ldots & x_N^{N-2} \\ T_{N-1}(x_1) & T_{N-1}(x_2) & \ldots & T_{N-1}(x_N) \end{pmatrix}
\end{aligned}$$

by adding appropriate multiples of the first $N-1$ rows to the last row. Now we do the same thing to all of the rows, except the first which we leave alone, working our way from the bottom to the top. In this way, we find that

$$\Delta(\cos\theta_1, \ldots, \cos\theta_N) = 2^{-(N-1)(N-2)/2} \det_{N\times N}(T_{j-1}(\cos\theta_k)). \tag{14}$$

For the Haar measure on $\mathrm{SO}(2N)$, we have

$$\begin{aligned}
dX_{SO(2N)} &= \frac{2^{(N-1)^2}}{\pi^N N!}\Delta(\cos\theta_1, \ldots, \cos\theta_N)^2 d\theta_1 \ldots d\theta_N \\
&= \frac{2^{N-1}}{\pi^N N!}\det_{N\times N}\big(T_{j-1}(\cos\theta_k)\big)^2 d\theta_1 \ldots d\theta_N.
\end{aligned}$$

If we multiply each row except the first by $\sqrt{2}$ we find that

$$\Delta(\cos\theta_1, \ldots, \cos\theta_N) = 2^{-(N-1)^2/2} \det_{N\times N}(T^*_{j-1}(\cos\theta_k))$$

where we let $T^*_j = \sqrt{2}T_j$ for $j \geq 1$ and $T^*_0 = T_0 = 1$. Then,

$$dX_{SO(2N)} = \frac{1}{\pi^N N!} \det_{N\times N}\left(T^*_{j-1}(\cos\theta_k)\right)^2 d\theta_1 \ldots d\theta_N. \tag{15}$$

By the Transposing Lemma,

$$\Delta(\cos\theta_1, \ldots, \cos\theta_N)^2 = 2^{-(N-1)^2} \det_{N\times N}\left(1 + 2\sum_{n=1}^{N-1} \cos n\theta_j \cos n\theta_k\right) \tag{16}$$

Recall that $S_N(\theta) = \frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}$ and $S(x) = \frac{\sin \pi x}{\pi x}$. Now

$$\begin{aligned}\sum_{n=-N}^{N} \cos nx &= \Re\sum_{n=-N}^{N} e^{inx} = \Re\frac{e^{i(N+1)x} - e^{-iNx}}{e^{ix} - 1} \\ &= \Re\frac{e^{i(N+1/2)x} - e^{-i(N+1/2)x}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin(N+1/2)x}{\sin x/2} = S_{2N+1}(x).\end{aligned}$$

Consequently,

$$\sum_{n=1}^{N} \cos nx = \frac{S_{2N+1}(x) - 1}{2}$$

so that

$$\begin{aligned}1 + 2\sum_{n=1}^{N-1} \cos nx \cos ny &= 1 + \sum_{n=1}^{N-1}\left(\cos n(x-y) + \cos n(x+y)\right) \\ &= \frac{S_{2N-1}(x-y) + S_{2N-1}(x+y)}{2}.\end{aligned}$$

Consequently, a basic identity is

$$2^{(N-1)^2} \prod_{1\leq j<k\leq N} (\cos\theta_k - \cos\theta_j)^2 = \det_{N\times N}\left(\frac{S_{2N-1}(\theta_k - \theta_j) + S_{2N-1}(\theta_k + \theta_j)}{2}\right)$$

from which we deduce by (15) that

$$\begin{aligned}
dX_{SO(2N)} &= \frac{1}{\pi^N N!} \det_{N\times N} \big(\frac{S_{2N-1}(\theta_k-\theta_j)+S_{2N-1}(\theta_k+\theta_j)}{2}\big)\ d\theta_1\ldots d\theta_N \\
&= \frac{1}{\pi^N N!} \det_{N\times N} \big(K_{SO(2N)}(\theta_j,\theta_k)\big)\ d\theta_1\ldots d\theta_N, \qquad (17)
\end{aligned}$$

where we define

$$K_{SO(2N)}(x,y) = \frac{S_{2N-1}(y-x)+S_{2N-1}(y+x)}{2}$$

and, for use in a moment,

$$K_{USp(2N)}(x,y) = \frac{S_{2N+1}(y-x)-S_{2N+1}(y+x)}{2}$$

and

$$K_{SO(2N+1)}(x,y) = \frac{S_{2N}(y-x)-S_{2N}(y+x)}{2}.$$

Now we do the same for the Haar measure of the symplectic group. Again, by elementary row operations on the determinant $\Delta(x_1,\ldots,x_N)$, we find that (recall that the leading coefficient of the Chebyshev polynomial $U_N(x)$ is $(2x)^N$),

$$\Delta(x_1,\ldots,x_N) = 2^{-N(N-1)/2} \det_{N\times N}(U_{j-1}(x_k)).$$

Then $dX_{USp(2N)}$, the Haar measure on USp$(2N)$, satisfies

$$\begin{aligned}
dX_{USp(2N)} &= \frac{2^{N^2}}{\pi^N N!}\Delta(\cos\theta_1,\ldots,\cos\theta_N)^2 \prod_{n=1}^{N} \sin^2\theta_n d\theta_1\ldots d\theta_N \\
&= \frac{2^N}{\pi^N N!} \det_{N\times N}\big(U_{j-1}(\cos\theta_k)\big)^2 \prod_{n=1}^{N} \sin^2\theta_n d\theta_1\ldots d\theta_N.
\end{aligned}$$

Now, by the Transposing Lemma

$$\begin{aligned}
\prod_{1\le j<k\le N}(\cos\theta_k-\cos\theta_j)^2 \prod_{n=1}^{N}\sin^2\theta_n &= 2^{-N(N-1)} \det_{N\times N}\left(\sum_{n=1}^{N}\sin n\theta_j \sin n\theta_k\right) \\
&= 2^{-N(N-1)} \det_{N\times N}\left(\frac{S_{2N+1}(\theta_k-\theta_j)-S_{2N+1}(\theta_k+\theta_j)}{2}\right)
\end{aligned}$$

since

$$\begin{aligned} 2\sum_{n=1}^{N}\sin nx \sin ny &= \sum_{n=1}^{N}\big(\cos n(x-y)-\cos n(x+y)\big) \\ &= \frac{S_{2N+1}(x-y)-S_{2N+1}(x+y)}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} dX_{USp(2N)} &= \frac{1}{\pi^N N!}\det_{N\times N}\Big(\frac{S_{2N+1}(\theta_k-\theta_j)-S_{2N+1}(\theta_k+\theta_j)}{2}\Big)\ d\theta_1\dots d\theta_N \\ &= \frac{1}{\pi^N N!}\det_{N\times N}\big(K_{USp(2N)}(\theta_k,\theta_j)\big)\ d\theta_1\dots d\theta_N. \end{aligned} \tag{18}$$

Finally, $dX_{SO(2N+1)}$, the Haar measure on $SO(2N+1)$, satisfies

$$\begin{aligned} dX_{SO(2N+1)} &= \frac{2^{N^2}}{\pi^N N!}\Delta(\cos\theta_1,\dots,\cos\theta_N)^2\prod_{n=1}^{N}\sin^2\frac{\theta_n}{2}d\theta_1\dots d\theta_N \\ &= \frac{2^N}{\pi^N N!}\det_{N\times N}\big(V_{j-1}(\cos\theta_k)\big)^2\prod_{n=1}^{N}\sin^2\frac{\theta_n}{2}d\theta_1\dots d\theta_N. \end{aligned}$$

By the Transposing Lemma,

$$\begin{aligned} dX_{SO(2N+1)} &= \frac{2^N}{\pi^N N!}\det_{N\times N}\Big(\sum_{n=1}^{N}\sin(n-\tfrac{1}{2})\theta_j\sin(n-\tfrac{1}{2})\theta_k\Big)d\theta_1\dots d\theta_N \\ &= \frac{1}{\pi^N N!}\det_{N\times N}\Big(\frac{S_{2N}(\theta_k-\theta_j)-S_{2N}(\theta_k+\theta_j)}{2}\theta_k\Big)d\theta_1\dots d\theta_N. \end{aligned}$$

Therefore,

$$dX_{SO(2N+1)} = \frac{1}{\pi^N N!}\det_{N\times N}\big(K_{SO(2N+1)}(\theta_k,\theta_j)\big)\ d\theta_1\dots d\theta_N. \tag{19}$$

5 Andréief's identity

As a consistency check, we now deduce that our measures have total mass one. To do this we use a formula of Andréief:

Lemma 2 (Andréief's identity) *For any interval J and integrable functions ϕ_j and ψ_j:*

$$\frac{1}{N!}\int_{J^N} \det_{N\times N}(\phi_j(\theta_k)) \det_{N\times N}(\psi_j(\theta_k))\ d\theta_1 \dots d\theta_N = \det_{N\times N}\Big(\int_J \phi_j(\theta)\psi_k(\theta)\ d\theta\Big). \tag{20}$$

5.1 Proof of Andréief's identity

We use the definition of determinant for a matrix $X = (x_{jk})$:

$$\det X = \sum_{\sigma\in\pi_N} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} x_{j,\sigma j}$$

where π_N denotes the collection of the $N!$ permutations of $[1, N] := \{1, 2, \ldots, N\}$. Thus,

$$
\begin{aligned}
&\int_{J^N} \det_{N\times N} \big(\phi_j(\theta_k)\big) \det_{N\times N} \big(\psi_j(\theta_k)\big) d\theta_1 \ldots \ d\theta_N \\
&= \int_{J^N} \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \sum_{\tau} \operatorname{sgn}(\tau) \prod_{k=1}^{N} \psi_k(\theta_{\tau k}) \ \prod_{i=1}^{N} d\theta_i \\
&\underset{\tau \to \sigma\tau}{=} \int_{J^N} \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \prod_{k=1}^{N} \psi_k(\theta_{\sigma\tau k}) \ \prod_{i=1}^{N} d\theta_i \\
&\underset{k \to \tau^{-1}k}{=} \int_{J^N} \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \prod_{k=1}^{N} \psi_{\tau^{-1}k}(\theta_{\sigma k}) \ \prod_{i=1}^{N} d\theta_i \\
&= \int_{J^N} \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \psi_{\tau^{-1}j}(\theta_{\sigma j}) \ \prod_{i=1}^{N} d\theta_i \\
&\underset{\tau \to \tau^{-1}}{=} \int_{J^N} \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \psi_{\tau j}(\theta_{\sigma j}) \ \prod_{i=1}^{N} d\theta_i \\
&= \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \int_J \phi_j(\theta) \psi_{\tau j}(\theta) \ d\theta \\
&= N! \sum_{\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \int_J \phi_j(\theta) \psi_{\tau j}(\theta) \ d\theta \\
&= \det_{N\times N} \Big(\int_J \phi_j(\theta) \psi_k(\theta) \ d\theta \Big).
\end{aligned}
$$

Note that virtually the same proof leads to the slightly more general result

$$
\begin{aligned}
\frac{1}{N!} \int_{J^N} \prod_{i=1}^{N} f(\theta_i) \det_{N\times N} (\phi_j(\theta_k)) \det_{N\times N} (\psi_j(\theta_k)) \ d\theta_1 \ldots d\theta_N \\
= \det_{N\times N} \Big(\int_J f(\theta) \phi_j(\theta) \psi_k(\theta) \ d\theta \Big).
\end{aligned}
$$

5.2 Verification that the Haar measures have total mass 1

Using

$$\phi_j(\theta) = e^{i(j-1)\theta}$$

we see that

$$\begin{aligned}
\int_{[0,2\pi]^N} dX_{U(N)} &= \int_{[0,2\pi]^N} |\det_{N\times N}(e^{i(j-1)\theta_k})|^2 \frac{d\theta_1 \ldots\ d\theta_N}{N!(2\pi)^N} \\
&= \frac{1}{(2\pi)^N} \det_{N\times N}\big(\int_0^{2\pi} e^{i(j-1)\theta} e^{-i(k-1)\theta}\ d\theta\big) \\
&= \frac{1}{(2\pi)^N} \det_{N\times N}\big(2\pi I\big) = 1.
\end{aligned}$$

This shows that the total mass of the Haar measure of $U(N)$ is 1.

We observe further that (15) and Andréief's identity together imply that

$$\int_{[0,\pi]^N} dX_{SO(2N)} = \frac{2^{N-1}}{\pi^N} \det_{N\times N}\left(\int_0^\pi T_{j-1}(\cos\theta) T_{k-1}(\cos\theta)\ d\theta\right) = 1,$$

since

$$\begin{aligned}
\int_0^\pi T_{j-1}(\cos\theta) T_{k-1}(\cos\theta)\ d\theta &= \int_0^\pi \cos(j-1)\theta \cos(k-1)\theta\ d\theta \\
&= \frac{1}{2}\int_0^\pi \big(\cos(j+k-2)\theta + \cos(j-k)\theta\big)\ d\theta \\
&= \frac{\pi}{2}\delta_{j,k}(1+\delta_{1,j})
\end{aligned}$$

because the integral is 0 unless $j=k$ in which case it is π if $j>1$ and 2π if $j=1$. Also,

$$\begin{aligned}
\int_{USp(2N)} dX &= \frac{2^N}{\pi^N} \det_{N\times N}\left(\int_0^\pi \sin^2\theta\ U_{j-1}(\cos\theta)\ U_{k-1}(\cos\theta)\ d\theta\right) \\
&= \frac{2^N}{\pi^N} \det_{N\times N}\left(\int_0^\pi \sin j\theta \sin k\theta\ d\theta\right) \\
&= \frac{2^N}{\pi^N} \det_{N\times N}\left(\frac{1}{2}\int_0^\pi \big(\cos(j-k)\theta - \cos(j+k)\theta\big)\ d\theta\right).
\end{aligned}$$

Since the integral is π when $j = k$ and 0 otherwise, this confirms that the total measure of $USp(2N)$ is 1.

Similarly, we can calculate that the total mass of $SO(2N+1)$ is 1.

6 Gaudin's Lemma

The following Lemma is the key to begin computing the statistics of interest.[1]

Lemma 3 (Gaudin's Lemma) *Suppose that we have a function f and a measurable set J such that*

$$\int_J f(x,\theta) f(\theta,y)\, d\theta = C f(x,y) \tag{21}$$

for all x and y where $C = C(J,f)$ is a constant. Suppose also that

$$\int_J f(x,x)\, dx = D, \tag{22}$$

where $D = D(J,f)$ is constant. Then

$$\int_J \det_{M\times M} \big(f(\theta_j,\theta_k)\big)\, d\theta_M = (D-(M-1)C) \det_{M-1} \big(f(\theta_j,\theta_k)\big). \tag{23}$$

This Lemma allows us to "integrate out" variables not under consideration when computing some statistic. We apply Gaudin's Lemma with $f(\theta) = S_N(\theta)$ and $J = [0, 2\pi]$, so that $D = S_N(0) = N$. Reëxpressing S_N as a geometric series and integrating term-by-term, we find that

$$\int_0^{2\pi} S_N(\theta_j - \theta) S_N(\theta - \theta_k)\, d\theta = 2\pi S_N(\theta_k - \theta_j),$$

so that $C = 2\pi$. Thus, for example,

$$\int_{[0,2\pi]} \det_{N\times N} \big(S_N(\theta_k - \theta_j)\big)\, d\theta_N = 2\pi \det_{(N-1)\times(N-1)} \big(S_N(\theta_k - \theta_j)\big).$$

[1] Editors' comment: This lemma is also applied in the lectures of Y.V. Fyodorov, page **??** Section 4.

Applying the Lemma repeatedly gives

$$\int_{[0,2\pi]^{N-n}} \det_{N\times N}\big(S_N(\theta_k-\theta_j)\big)\ d\theta_{n+1}\dots\ d\theta_N$$
$$= (N-n)!(2\pi)^{N-n}\det_{n\times n}\big(S_N(\theta_k-\theta_j)\big).$$

In particular,

$$\int_{U(N)} \sum_{\substack{J\subset\{1,\dots,N\}\\ J=\{j_1,\dots,j_n\}}} f(\theta_{j_1},\dots,\theta_{j_n})dX_{U(N)}$$
$$= \frac{1}{(2\pi)^n n!}\int_{[0,2\pi]^n} f(\theta_1,\dots,\theta_n)\det_{n\times n} S_N(\theta_k-\theta_j)\ d\theta_1\dots d\theta_n. \quad (24)$$

Proof of Gaudin's Lemma. Let π_M be the symmetric group on $\{1,\dots,M\}$. Then,

$$\det_{M\times M}\big(f(\theta_j,\theta_k)\big) \;=\; \sum_{\sigma\in\pi_M}\operatorname{sgn}(\sigma)\prod_{j=1}^{M} f(\theta_j,\theta_{\sigma j}).$$

If $\sigma M \neq M$, then

$$\begin{aligned}\int_J \prod_{j=1}^{M} f(\theta_j,\theta_{\sigma j})d\theta_M &= \prod_{\substack{j=1\\ \sigma j\neq M}}^{M-1} f(\theta_j,\theta_{\sigma j})\int_J f(\theta_{\sigma^{-1}M},\theta_M)f(\theta_M,\theta_{\sigma M})\ d\theta_M \\ &= f(\theta_{\sigma^{-1}M},\theta_{\sigma M})\prod_{\substack{j=1\\ \sigma j\neq M}}^{M-1} f(\theta_j,\theta_{\sigma j}). \quad (25)\end{aligned}$$

For a permutation $\sigma\in\pi_M$ with $\sigma M\neq M$ define a permutation $\sigma'\in\pi_{M-1}$ by

$$\sigma' j = \begin{cases} \sigma j & \text{if } \sigma j\neq M \\ \sigma M & \text{if } \sigma j = M \end{cases}.$$

Then (25) may be reëxpressed as

$$\int_J \prod_{j=1}^{M} f(\theta_j,\theta_{\sigma j})d\theta_M = C\prod_{j=1}^{M-1} f(\theta_j,\theta_{\sigma' j}).$$

Clearly, each permutation σ' arises from $(M-1)$ different σ. Note also that $\mathrm{sgn}(\sigma') = -\mathrm{sgn}(\sigma)$. Thus, we have

$$\begin{aligned}\int_J \sum_{\substack{\sigma\in\pi_M\\ \sigma M\neq M}} \mathrm{sgn}(\sigma)\prod_{j=1}^{M} f(\theta_j,\theta_{\sigma j})d\theta_M &= -(M-1)C\sum_{\sigma'\in\pi_{M-1}}\mathrm{sgn}(\sigma')\prod_{j=1}^{M-1} f(\theta_j,\theta_{\sigma' j})\\ &= -(M-1)C\det_{M-1}\big(f(\theta_j,\theta_k)\big).\end{aligned}$$

Now consider the σ for which $\sigma M = M$; now let σ' be defined by $\sigma' j = \sigma j$ for $j \leq M-1$. Then, for these σ, we have

$$\begin{aligned}\int_J \prod_{j=1}^{M} f(\theta_j,\theta_{\sigma j})d\theta_M &= \prod_{j=1}^{M-1} f(\theta_j,\theta_{\sigma j})\int_J f(\theta_m,\theta_M)\ d\theta_M\\ &= D\prod_{j=1}^{M-1} f(\theta_j,\theta_{\sigma' j}).\end{aligned}$$

These σ' have the same sign as the σ they came from. Therefore,

$$\begin{aligned}\int_J \sum_{\substack{\sigma\in\pi_M\\ \sigma M=M}} \mathrm{sgn}(\sigma)\prod_{j=1}^{M} f(\theta_j,\theta_{\sigma j})d\theta_M &= D\sum_{\sigma'\in\pi_{M-1}}\mathrm{sgn}(\sigma')\prod_{j=1}^{M-1} f(\theta_j,\theta_{\sigma' j})\\ &= D\det_{M-1}\big(f(\theta_k,\theta_j)\big).\end{aligned}$$

Combining the two cases we obtain the Lemma.

6.1 Calculation for orthogonal and symplectic cases

Recall that

$$K_{SO(2N)}(x,y) = \frac{S_{2N-1}(x-y)+S_{2N-1}(x+y)}{2} = 1 + 2\sum_{n=1}^{N-1}\cos nx\cos ny$$

and

$$K_{USp(2N)}(x,y) = \frac{S_{2N+1}(x-y)-S_{2N+1}(x+y)}{2} = 2\sum_{n=1}^{N}\sin nx\sin ny.$$

Then, Gaudin's Lemma for these groups is expressed as

$$\int_{[0,\pi]^{N-n}} \det_{N\times N}\left(K_{G(N)}(\theta_j,\theta_k)\right) d\theta_{n+1}\ldots\, d\theta_N = (N-n)!\pi^{N-n} \det_{n\times n}\left(K_{G(N)}(\theta_j,\theta_k)\right)$$

where $G(N)$ can stand for $USp(2N), SO(2N)$, or $SO(2N+1)$. This allows us to "integrate out" variables in the orthogonal and symplectic settings.

Note, for future reference, that

$$\lim_{N\to\infty}\frac{1}{2N}S_{2N+1}(\pi x/N) = \frac{\sin\frac{(N+1/2)\pi x}{N}}{2N\sin\frac{\pi x}{2N}} = \frac{\sin\pi x}{\pi x} = S(x)$$

so that

$$K_G(x,y) := \lim_{N\to\infty}\frac{K_{G(N)}(\pi x/N, \pi y/N)}{2N} = S(y-x)\pm S(y+x).$$

To prove Gaudin's Lemma in this situation, it again suffices to prove the $n = N-1$ case, since the general case follows by a repeated application of this case:

$$\int_{[0,\pi]} \det_{N\times N}\left(K_{G(N)}(\theta_j,\theta_k)\right)\, d\theta_N = \pi \det_{(N-1)\times(N-1)}\left(K_{G(N)}(\theta_j,\theta_k)\right).$$

The key formulae are

$$\int_0^\pi K_{G(N)}(\theta_j,\theta)K_{G(N)}(\theta,\theta_k)\, d\theta = \pi K_{G(N)}(\theta_j,\theta_k).$$

Knowing this, the rest of the proof is the same; so we now verify these formulae. We calculate

$$\begin{aligned}
&\int_0^\pi K_{USp(2N)}(x,\theta)K_{USp(2N)}(\theta,y)\, d\theta\\
&= \int_0^\pi \sum_{m=1}^N 2\sin mx\sin m\theta \sum_{n=1}^N 2\sin n\theta\sin ny\, d\theta\\
&= 4\sum_{m,n=1}^N \sin mx\sin ny\int_0^\pi \sin m\theta\sin n\theta\, d\theta\\
&= 2\sum_{m,n=1}^N \sin mx\sin ny\int_0^\pi (\cos(m-n)\theta - \cos(m+n)\theta)\, d\theta\\
&= 2\pi\sum_{n=1}^N \sin nx\sin ny = \pi K_{USp(2N)}(x,y).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^\pi K_{SO(2N)}(x,\theta)K_{SO(2N)}(\theta,y)\ d\theta \\
&\quad = \int_0^\pi (1+2\sum_{m=1}^{N-1}\cos mx\cos m\theta)(1+2\sum_{n=1}^{N-1}\cos n\theta\cos ny)\ d\theta \\
&\quad = \pi + 4\sum_{m,n=1}^{N-1}\cos mx\cos ny\int_0^\pi \cos m\theta\cos n\theta\ d\theta \\
&\quad = \pi + 2\sum_{m,n=1}^{N-1}\cos mx\cos ny\int_0^\pi (\cos(m-n)\theta+\cos(m+n)\theta)\ d\theta \\
&\quad = \pi(1+2\sum_{n=1}^{N-1}\cos nx\cos ny) = \pi K_{SO(2N)}(x,y).
\end{aligned}$$

Similarly for $K_{SO(2N+1)}$.

7 n-level density

7.1 Unitary

We can use Gaudin's Lemma to compute an integral of the sort

$$\int_{U(N)}\sum_{j=1}^N f(\theta_j)dX,$$

or

$$\int_{U(N)}\sum_{1\le j<k\le N} f(\theta_j,\theta_k)dX,$$

or

$$\int_{U(N)}\sum_{1\le j_1<\cdots<j_n\le N} f(\theta_{j_1},\ldots,\theta_{j_n})dX.$$

With an obvious notation, we write the last integral as

$$\int_{U(N)}\sum_{\substack{B\subset[1,N]\\ |B|=n}} f_B(\theta)dX$$

These are precisely the definitions of the 1-, 2-, and n-level densities. By Gaudin's Lemma, these integrals are, respectively,

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\ d\theta,$$

$$\frac{1}{2!(2\pi)^2}\int_{[0,2\pi]^2} f(\theta_1,\theta_2)\det_{2\times 2} S_N(\theta_k-\theta_j)\ d\theta_1\ d\theta_2,$$

and [2]

$$\frac{1}{n!(2\pi)^n}\int_{[0,2\pi]^n} f(\theta_1,\ldots,\theta_n)\det_{n\times n} S_N(\theta_k-\theta_j)\ d\theta_1\ldots\ d\theta_n. \tag{26}$$

7.2 Normalized eigenangles and large N limits

For a matrix $X\in U(N)$ with eigenvalues

$$e^{i\theta_1},\ldots,e^{i\theta_N}$$

we let

$$\tilde{\theta}_j=\theta\frac{N}{2\pi} \tag{27}$$

be the normalized eigenangles. They satisfy

$$0\le\tilde{\theta}_1\le\cdots\le\tilde{\theta}_N<N.$$

The sequence of $\tilde{\theta}$ have mean spacing 1 and so give a way to compare statistics for different N. Thus, for the n-level density, we have (for a rapidly decaying smooth f)

$$\begin{aligned}
&\lim_{N\to\infty}\int_{U(N)}\sum_{\substack{B\subset[1,N]\\|B|=n}} f_B(\tilde{\theta})dX\\
&=\lim_{N\to\infty}\frac{1}{n!(2\pi)^n}\int_{[0,2\pi]^n} f(\tilde{\theta}_1,\ldots,\tilde{\theta}_n)\det_{n\times n} S_N(\theta_k-\theta_j)\ d\theta_1\ldots\ d\theta_n\\
&=\lim_{N\to\infty}\frac{1}{n!}\int_{[0,N]^n} f(x_1,\ldots,x_n)\det_{n\times n}\frac{1}{N}S_N\big(\frac{2\pi(x_k-x_j)}{N}\big)\ dx_1\ldots\ dx_n\\
&=\frac{1}{n!}\int_{\mathbf{R}_+^n} f(x_1,\ldots,x_n)\det_{n\times n} S(x_k-x_j)\ dx_1\ldots dx_n.
\end{aligned} \tag{28}$$

[2]Editors' comment: Up to a constant factor $f(\theta_1,\ldots,\theta_n)$ is being integrated against the quantity defined in the lectures of Y.V. Fyodorov, page **??**, Section 3, as the n-point correlation function.

We say that $\det_n S(x_k - x_j)$ is the n-level density function for U.

7.3 Orthogonal and symplectic

We can use Gaudin's Lemma to compute

$$\int_{SO(2N)} \sum_{\substack{B\subset[1,N]\\|B|=n}} f_B(\theta) dX$$

and

$$\int_{USp(2N)} \sum_{\substack{B\subset[1,N]\\|B|=n}} f_B(\theta) dX.$$

By Weyl's integration formula and Gaudin's Lemma, these integrals are

$$\frac{1}{n!\pi^n} \int_{[0,\pi]^n} f(\theta_1,\ldots,\theta_n) \det_{n\times n}\left(K_{SO(2N)}(\theta_k,\theta_j)\right) d\theta_1 \ldots\ d\theta_n$$

and

$$\frac{1}{n!\pi^n} \int_{[0,\pi]^n} f(\theta_1,\ldots,\theta_n) \det_{n\times n}\left(K_{USp(2N)}(\theta_k,\theta_j)\right) d\theta_1 \ldots\ d\theta_n$$

respectively.

For eigenangles of matrices in $SO(2N)$ and $USp(2N)$ we let

$$\tilde{\theta}_j = \theta \frac{N}{\pi}$$

be the normalized eigenangles. Thus, for the n-level density, we have (for a rapidly decaying smooth f)

$$\begin{aligned}
&\lim_{N\to\infty} \int_{SO(2N)} \sum_{1\le j_1<\cdots<j_n\le N} f(\tilde{\theta}_{j_1},\ldots,\tilde{\theta}_{j_n}) dX \\
&\quad = \lim_{N\to\infty} \frac{1}{n!\pi^n} \int_{[0,\pi]^n} f(\tilde{\theta}_1,\ldots,\tilde{\theta}_n) \det_{n\times n}\left(K_{SO(2N)}(\theta_k,\theta_j)\right) d\theta_1 \ldots\ d\theta_n \\
&\quad = \frac{1}{n!} \int_{\mathbf{R}_+^n} f(\theta_1,\ldots,\theta_n) \det_{n\times n}\left(K_{SO}(\theta_k,\theta_j)\right) d\theta_1 \ldots d\theta_n \qquad (29)
\end{aligned}$$

for the n-level density for SO even and

$$\lim_{N\to\infty}\int_{USp(2N)}\sum_{1\le j_1<\cdots<j_n\le N} f(\tilde{\theta}_{j_1},\ldots,\tilde{\theta}_{j_n})dX$$
$$= \lim_{N\to\infty}\frac{1}{n!\pi^n}\int_{[0,\pi]^n} f(\tilde{\theta}_1,\ldots,\tilde{\theta}_n)\det_{n\times n}\left(K_{USp(2N)}(\theta_k,\theta_j)\right)\,d\theta_1\ldots\,d\theta_n$$
$$= \frac{1}{n!}\int_{\mathbf{R}_+^n} f(\theta_1,\ldots,\theta_n)\det_{n\times n}\left(K_{USp}(\theta_k,\theta_j)\right)\,d\theta_1\ldots d\theta_n \qquad (30)$$

for the n-level density for USp. The n-level density for $SO(2N+1)$ is slightly complicated by the fact that the matrices in this ensemble always have an eigenangle equal to 0. This fact leads to the presence of a δ-function in the formulation of the n-level density function.

8 Correlations

8.1 Pair correlation for $U(N)$

Let f be a suitable test function and consider

$$Q_N(f) := \int_{U(N)}\sum_{j<k} f(\tilde{\theta}_j-\tilde{\theta}_k)dX.$$

Applying Gaudin's Lemma we find that

$$Q_N(f) = \int_{[0,2\pi]^2} f(\tilde{\theta}_1-\tilde{\theta}_2)\det\begin{pmatrix} N & S_N(\theta_1-\theta_2) \\ S_N(\theta_1-\theta_2) & N\end{pmatrix}\frac{d\theta_1\,d\theta_2}{2(2\pi)^2}.$$

After a change of variables, this is

$$= \frac{1}{2}\int_{[0,N]^2} f(\theta_1-\theta_2)\det_{2\times 2}\frac{1}{N}S_N\big(\frac{2\pi(\theta_k-\theta_j)}{N}\big)\,d\theta_1\,d\theta_2.$$

After expanding the determinant and performing another change of variables, we have

$$Q_N(f) = \frac{1}{2}\int_{[-N,N]} f(v)\left(1-\left(\frac{S_N(2\pi v/N)}{N}\right)^2\right)(N-|v|)\,dv.$$

Now

$$\lim_{N\to\infty} \frac{1}{N} S_N\big(\frac{2\pi v}{N}\big) = \lim_{N\to\infty} \frac{1}{N} \frac{\sin \pi v}{\sin \frac{\pi v}{N}} = \frac{\sin \pi v}{\pi v} = S(v).$$

Now it follows, with a little bit of analysis, that

$$\lim_{N\to\infty} \frac{1}{N} Q_N(f) = \frac{1}{2} \int_{-\infty}^{\infty} f(v) \left(1 - \big(\frac{\sin \pi v}{\pi v}\big)^2\right) \, dv.$$

This is the same as the pair correlation for zeros of $\zeta(s)$ found by Montgomery[3]. (Note that the factor $\frac{1}{2}$ in front of our formula is because we defined out correlation sum to be over $j < k$ rather than $j \neq k$.) This important fact was fortuitously discovered at tea at the Institute for Advanced Study one afternoon in 1971 when Chowla introduced Hugh Montgomery and Freeman Dyson to each other.

8.2 n-correlation for $U(N)$

Let $f(\theta_1, \dots, \theta_n)$ be a test-function which is translation invariant i.e. $f(\theta_1 + t, \dots, \theta_n + t) = f(\theta_1, \dots, \theta_n)$. Let's suppose, for convenience, that $f(0, \theta_2, \dots, \theta_n)$ is compactly supported, say on $[0, A]$. We seek to evaluate

$$Q_N(f) = \int_{U(N)} \sum_{1\le j_1 < j_2 < \cdots < j_n \le N} f(\tilde{\theta}_{j_1}, \dots, \tilde{\theta}_{j_n}) dX.$$

By Gaudin's Lemma and a change of variables, this is

$$= \frac{1}{n!} \int_{[0,N]^n} f(\theta_1, \dots, \theta_n) \det_{n\times n} \frac{1}{N} S_N(2\pi(\theta_j - \theta_k)/N) d\theta_1 \dots d\theta_n$$

$$= \int_{0\le\theta_1\le\cdots\le\theta_n\le N} f(\theta_1, \dots, \theta_n) \det_{n\times n} \frac{1}{N} S_N(2\pi(\theta_j - \theta_k)/N) d\theta_1 \dots d\theta_n.$$

We make the change of variable $x_1 = \theta_1$, $x_2 = \theta_2 - \theta_1$, ... $x_n = \theta_n - \theta_1$ and the integral becomes

$$\int_0^N \int_{0\le x_2\le\cdots\le x_n\le N - x_1} g(x_1, x_2 + x_1, \dots, x_n + x_1) dx_2 \dots dx_n \, dx_1$$

[3] Editors' comment: See lectures by D.A. Goldston, page ??, equation 6.7.

where $g(x_1,\dots,x_n) = f(x_1,\dots,x_n)\det(\frac{S_N(2\pi(x_k-x_j)/N)}{N})$. Since g is translation invariant and $g(0,x_2,\dots,x_n)$ is compactly supported, for sufficiently large N this is

$$= \int_0^N \int_{0\le x_2\le\cdots\le x_n\le N-x_1} g(0,x_2,\dots,x_n)dx_2\dots dx_n\, dx_1$$
$$= \int_{0\le x_2\le\cdots\le x_n\le A} g(0,x_2,\dots,x_n)\int_0^{N-A} dx_1\, dx_2\dots dx_n.$$

Thus

$$\lim_{N\to\infty}\frac{Q_N(f)}{N} = \frac{1}{(n-1)!}\int_{\mathbf{R}_+^{n-1}} f(x_1,x_2,\dots,x_n)\det_{n\times n} S(x_j-x_k)\big|_{x_1=0}dx_2\dots dx_n. \tag{31}$$

8.3 n-correlation for orthogonal and symplectic

Let $f(\theta_1,\dots,\theta_n)$ be a test-function as in the last section. We seek to evaluate

$$Q_N(f) = \int_{USp(N)} \sum_{1\le j_1<j_2<\cdots<j_n\le N} f(\tilde\theta_{j_1},\dots,\tilde\theta_{j_n})dX.$$

By Gaudin's Lemma and a change of variables, this is

$$= \frac{1}{n!}\int_{[0,N]^n} f(\theta_1,\dots,\theta_n)\det_{n\times n}\frac{1}{2N}\Big(S_{2N-1}(\pi(\theta_k-\theta_j)/N) + S_{2N-1}(\pi(\theta_k+\theta_j)/N)\Big)d\theta_1\dots d\theta_n$$
$$= \int_{0\le\theta_1\le\cdots\le\theta_n\le N} f(\theta_1,\dots,\theta_n)\det_{n\times n}\frac{1}{2N}\Big(S_{2N-1}(\pi(\theta_k-\theta_j)/N) + S_{2N-1}(\pi(\theta_k+\theta_j)/N)\Big)d\theta_1\dots d\theta_n.$$

We make the change of variable $x_1=\theta_1$, $x_2=\theta_2-\theta_1$, ... $x_n=\theta_n-\theta_1$

and the integral becomes, for sufficiently large N,

$$\int_{0\le x_2\le\cdots\le x_n\le A} f(0,x_2,\ldots,x_n)\ \int_0^{N-x_n} \det_{n\times n} \frac{1}{2N}\Big(S_{2N-1}(\pi(x_k-x_j)/N)\big|_{x_1=0} + S_{2N-1}(\pi(x_k^*+x_j^*+2x_1)/N)\Big) dx_1\, dx_2 \ldots dx_n$$

where $x_j^* = x_j$ if $j\neq 1$ whereas $x_1^*=0$. Now we claim that

$$\lim_{N\to\infty}\frac{1}{N}\int_0^{N-x_n} \det_{n\times n}\frac{1}{2N}\big(S_{2N-1}(\pi(x_k-x_j)/N)\big|_{x_1=0} + S_{2N-1}(\pi(x_k^*+x_j^*+2x_1)/N)\big)dx_1 = \det_{n\times n} S(x_k-x_j)\big|_{x_1=0}.$$

To see this claim, note that in the expansion of the determinant there are $n!$ terms each of which is a product of n factors $\frac{1}{2N}S_{2N-1}(\pi(x_k-x_j))|_{x_1=0} + \frac{1}{2N}S_{2N-1}(\pi(x_k^*+x_j^*+2x_1))$. If we multiply out each term, there are 2^n terms, all but one of which will contain at least one factor with $\frac{1}{2N}S_{2N-1}(\pi(x_k + x_j + 2x_1))$. Any of the terms with at least one factor like this will tend to 0 after integrating with respect to x_1 and dividing by N; for letting

$$c(a,b,N)(x) = \frac{\sin(ax+b)}{N\sin(ax/N+b/N)},$$

it is not difficult to see that $c(a,b,N)(x)\le \frac{2}{\pi}\frac{\sin(ax+b)}{ax+b}$ provided that $ax+b < \frac{\pi N}{2}$, and $|c(a,b,N)(x)|\le 1$ for all x and integer N. Therefore, using the fact that $\int_0^B \left(\frac{\sin x}{x}\right)^j\, dx$ is uniformly bounded in B for each fixed j, we see that

$$\frac{1}{N}\int_0^{N-B}\prod_{j=1}^{J} c(a_j,b_j,N)(x)\ dx \to 0$$

as $N\to\infty$ through integers. This leaves only the term with all $\frac{1}{2N}S_{2N-1}(\pi(x_k - x_j))$ factors which tend to $S(x_k-x_j)$ as $N\to\infty$.

Thus, just as in the case of $U(N)$, we find that

$$\lim_{N\to\infty}\frac{Q_N(f)}{N} = \frac{1}{(n-1)!}\int_{\mathbf{R}_+^{n-1}} f(x_1,x_2,\ldots,x_n)\det_{n\times n} S(x_j-x_k)\big|_{x_1=0} dx_2\ldots dx_n.$$

In particular, the scaled limit of the n-correlation functions are the same for all of unitary, orthogonal, and symplectic groups.

9 Neighbor spacings

9.1 Nearest neighbor for $U(N)$

We derive the combinatorial relation between nearest neighbor spacings and n-correlations (see [KS], Lemma 2.3.8). For a sequence $Y : \theta_1 \leq \cdots \leq \theta_N$ let

$$\mathcal{S}_n(s, Y) := \#\{j : \theta_{j+n} - \theta_j \leq s\},$$

and

$$\mathcal{C}_m(s, Y) := \#\{B \subset \{1, \ldots, N\} : |B| = m, \max_{j,k \in B} |\theta_j - \theta_k| \leq s\}.$$

These are related to the Sep and Clump functions used in Katz-Sarnak.

Lemma 4 (Combinatorial Lemma) *For any Y,*

$$\mathcal{C}_{m+2}(s, Y) = \sum_{n \geq m} \binom{n}{m} \mathcal{S}_{n+1}(s, Y). \tag{32}$$

Proof. Take an $m+2$-tuple of indices $i_0 < i_1 < \cdots < i_{m+1}$ whose endpoints satisfy $\theta_{i_{m+1}} - \theta_{i_0} \leq s$. Let $n = i_{m+1} - i_0$ so that the pair of endpoints is counted in $\mathcal{S}_n(s, Y)$. Then there are $\binom{n-1}{m}$ sets of points of size m between these endpoints, which, taken with the endpoints can be counted in $\mathcal{C}_{m+2}(s, Y)$. Therefore, $\mathcal{C}_{m+2} = \sum \binom{n-1}{m} \mathcal{S}_n$. Adjusting the index n by one gives the result.

In general, the relation $a_m = \sum_{n \geq m} \binom{n}{m} b_n$ can be inverted to give

$$b_m = \sum_{n \geq m} (-1)^{n-m} \binom{n}{m} a_n.$$

This follows from the identity for binomial coefficients

$$\sum_{\ell=m}^{n} (-1)^{\ell} \binom{\ell}{m} \binom{n}{\ell} = \begin{cases} (-1)^m \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}.$$

Thus, with $a_m = \mathcal{C}_{m+2}$ and $b_n = \mathcal{S}_{n+1}$ we have

Corollary 1

$$\mathcal{S}_{m+1}(s,Y) = \sum_{n\geq m} (-1)^{n-m} \binom{n}{m} \mathcal{C}_{n+2}(s,Y) \tag{33}$$

or, after adjusting the indices,

$$\mathcal{S}_{m}(s,Y) = \sum_{n\geq m} (-1)^{n-m-1} \binom{n-2}{m-1} \mathcal{C}_{n}(s,Y). \tag{34}$$

Let $\tilde{Y}_X$ be the sequence of normalized eigenangles of X. We want to compute

$$\begin{aligned}\int_0^s \mu_1(x)\,dx := {} & \text{Prob}\,\{\text{Neighboring eigenangles are} < s \text{ apart}\} \\ = {} & \lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \mathcal{S}_1(s,\tilde{Y}_X)dX\end{aligned}$$

and more generally

$$\begin{aligned}\int_0^s \mu_m(x)\,dx := {} & \text{Prob}\,\{m\text{th neighboring eigenangles are} < s \text{ apart}\} \\ = {} & \lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \mathcal{S}_m(s,\tilde{Y}_X)dX.\end{aligned}$$

Applying the n-correlation calculation with the translation invariant function

$$f(\theta_1,\ldots,\theta_n) = \prod_{1\leq j<k\leq n} \chi_{[0,s]}(|\theta_j-\theta_k|)$$

gives,

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \mathcal{C}_n(s,\tilde{Y}_X)dX &= \lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \sum_{\substack{B\subset\{1,\ldots,N\}\\ |B|=n}} f(\theta_B)dX \\ &= \frac{1}{(n-1)!} \int_{[0,s]^{n-1}} \det_{n\times n} S(x_j-x_k)\big|_{x_1=0} dx_2\ldots dx_n.\end{aligned}$$

Thus, by (33)

$$\int_0^s \mu_m(x)\ dx = \sum_{n=m+1}^{\infty} \frac{(-1)^{n-m-1}}{(n-1)!} \binom{n-2}{m-1} \\ \times \int_{[0,s]^{n-1}} \det_{n\times n} S(x_j - x_k)\big|_{x_1=0} dx_2 \dots dx_n.$$

In particular, for the nearest neighbor spacing, we have

$$\mu_1(s) = \frac{d}{ds} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)!} \int_{[0,s]^{n-1}} \det_{n\times n} S(x_j - x_k)\big|_{x_1=0} dx_2 \dots dx_n.$$

Now, for any symmetric, even, translation invariant function g,

$$\frac{d}{ds} \int_{[0,s]^m} g(x_1, \dots, x_m) dx_1 \dots dx_m \quad = \quad m \int_{[0,s]^{m-1}} g(s, x_2, \dots, x_m) dx_2 \dots dx_m.$$

Therefore,

$$\begin{aligned} \mu_1(s) &= \frac{d^2}{ds^2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \int_{[0,s]^n} \det_{n\times n} S(x_j - x_k)\ dx_1 dx_2 \dots dx_n \\ &= \frac{d^2}{ds^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{[0,s]^n} \det_{n\times n} S(x_j - x_k)\ dx_1 dx_2 \dots dx_n. \end{aligned}$$

Also, temporarily letting

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{[0,s]^n} \det_{n\times n} S(x_j - x_k)\ dx_1 dx_2 \dots dx_n,$$

we have

$$\begin{aligned} \mu_m(s) &= \frac{d^2}{ds^2} \frac{d^{m-1}}{dz^{m-1}} \left(\frac{F(z) - 1 - z \int_0^s \det_{1\times 1} S\ d\theta}{z^2} \right)\Bigg|_{z=-1} \\ &= \frac{d^2}{ds^2} \frac{d^{m-1}}{dz^{m-1}} \frac{F(z)}{z^2}\Bigg|_{z=-1}. \end{aligned} \tag{35}$$

In the next few sections we will work toward relating the right side of this formula to another simple function.

9.2 Gram's identity

An identity of Gram is helpful for our further considerations.

Lemma 5 (Gram's Identity) *For an interval J and integrable functions ϕ_j and ψ_j,*

$$\det_{N\times N}(I+z\int_J \phi_j(x)\psi_k(x)\,dx) = \sum_{n=0}^{N} \frac{z^n}{n!}\int_{J^n} \big(\det_{n\times n}\sum_{h=1}^{N}\phi_h(x_j)\psi_h(x_k)\big)\,dx_1\dots dx_n. \tag{36}$$

Proof . The left-hand-side of (36) is

$$\begin{aligned}
&\det_{N\times N}(I+z\int_J \phi_j(x)\psi_k(x)\,dx)\\
&\quad= \sum_{\sigma\in\pi_N}\operatorname{sgn}(\sigma)\prod_{j=1}^{N}\big(\delta_{j,\sigma j}+z\int_J\phi_j(x)\psi_{\sigma j}(x)\,dx\big)\\
&\quad= \sum_{\sigma\in\pi_N}\operatorname{sgn}(\sigma)\sum_{A\subset N}\prod_{j\notin U}\delta_{j,\sigma j}\prod_{j\in U} z\int_J\phi_j(x)\psi_{\sigma j}(x)\,dx\\
&\quad= \sum_{A\subset N} z^{|A|}\sum_{\sigma\in\pi_A}\operatorname{sgn}(\sigma)\prod_{j\in A}\int_J\phi_j(x)\psi_{\sigma j}(x)\,dx\\
&\quad= \sum_{n=0}^{N} z^n \sum_{\substack{A\subset N\\ |A|=n}}\det_A\int_J\phi_j(x)\psi_k(x)\,dx
\end{aligned}$$

and the right-hand-side is

$$\begin{aligned}
&= \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \sum_{\sigma\in\pi_n} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} \sum_{h=1}^{N} \phi_h(x_j)\psi_h(x_{\sigma j})\ dx_1 \ldots dx_N \\
&= \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \sum_{\sigma\in\pi_n} \mathrm{sgn}(\sigma) \sum_{\lambda:[1,n]\to[1,N]} \prod_{j=1}^{n} \phi_{\lambda j}(x_j)\psi_{\lambda j}(x_{\sigma j})\ dx_1 \ldots dx_N \\
&= \sum_{n=0}^{N} \frac{z^n}{n!} \sum_{\sigma\in\pi_n} \mathrm{sgn}(\sigma) \sum_{\lambda:[1,n]\to[1,N]} \prod_{j=1}^{n} \int_J \phi_{\lambda j}(x)\psi_{\lambda\sigma^{-1}j}(x)\ dx \\
&= \sum_{n=0}^{N} \frac{z^n}{n!} \sum_{\lambda:[1,n]\to[1,N]} \sum_{\sigma\in\pi_n} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} \int_J \phi_{\lambda j}(x)\psi_{\lambda\sigma^{-1}j}(x)\ dx \\
&= \sum_{n=0}^{N} \frac{z^n}{n!} \sum_{\lambda:[1,n]\to[1,N]} \det\Big(\int_J \phi_{\lambda j}(x)\psi_{\lambda k}(x)\ dx\Big).
\end{aligned}$$

If λ is not one-to-one, then the inner determinant is 0. If λ is one-to-one, call the image A. Each such set A appears $n!$ times and we get the left-hand-side.

Remark. This proof is reminiscent of the proof that that the determinant of a product is the product of the determinants. Thus,

$$\begin{aligned}
\det_{N\times N}(AB) &= \det(a_{jh})(b_{hk}) = \det(\sum_{h=1}^{N} a_{jh}b_{hk}) \\
&= \sum_{\sigma\in\pi_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} \sum_{h=1}^{N} a_{jh}b_{h,\sigma j} \\
&= \sum_{\sigma\in\pi_N} \mathrm{sgn}(\sigma) \sum_{\lambda:[1,N]\to[1,N]} \prod_{j=1}^{N} a_{j,\lambda j}b_{\lambda j,\sigma j}
\end{aligned}$$

where the sum over λ is over all of the N^N functions from $[1,N]$ to itself. Now, we claim that for each fixed λ the sum

$$\sum_{\sigma\in\pi_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} a_{j,\lambda j}b_{\lambda j,\sigma j}$$

is 0 unless λ is actually a permutation. For suppose that $\lambda u = \lambda v$ for some $u \neq v \in [1, N]$. Then, for each σ let σ' be the permutation defined by $\sigma' j = \sigma j$ if $j \neq u, v$ whereas

$$\sigma' j = \begin{cases} \sigma u \text{ if } j = v \\ \sigma v \text{ if } j = u \end{cases}.$$

In this way π_N splits up into pairs (σ, σ') of permutations. Note that $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\sigma')$. Then

$$\begin{aligned} \prod_{j=1}^{N} a_{j,\lambda j} b_{\lambda j,\sigma j} &= a_{u,\lambda u} b_{\lambda u,\sigma u} a_{v,\lambda v} b_{\lambda v,\sigma v} \prod_{j \neq u,v} a_{j,\lambda j} b_{\lambda j,\sigma j} \\ &= a_{u,\lambda u} b_{\lambda v,\sigma' v} a_{v,\lambda v} b_{\lambda u,\sigma' u} \prod_{j \neq u,v} a_{j,\lambda j} b_{\lambda j,\sigma' j} \end{aligned}$$

.

Thus, the contribution from σ cancels that from σ' in the case that $\lambda u = \lambda v$. Therefore,

$$\begin{aligned} \det(AB) &= \sum_{\sigma,\tau \in \pi_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} a_{j,\tau j} b_{\tau j,\sigma j} \\ &= \sum_{\sigma,\tau \in \pi_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} a_{j,\tau j} \prod_{k=1}^{N} b_{\tau k,\sigma k} \\ &\underset{\sigma \to \sigma\tau}{=} \sum_{\sigma,\tau \in \pi_N} \mathrm{sgn}(\sigma\tau) \prod_{j=1}^{N} a_{j,\tau j} \prod_{k=1}^{N} b_{\tau k,\sigma\tau k} \\ &\underset{k \to \tau^{-1} k}{=} \sum_{\sigma,\tau \in \pi_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \prod_{j=1}^{N} a_{j,\tau j} \prod_{k=1}^{N} b_{k,\sigma k} \\ &= \det A \det B. \end{aligned}$$

9.3 Intervals with precisely n eigenvalues

Let $E_{G(N)}(n, J)$ be the measure of the set of matrices $A \in G(N)$ which have precisely n eigenvalues in the interval J. Here $G(N)$ can be $U(N)$, $SO(2N)$, $SO(2N+1)$, or $USp(2N)$; we denote the Haar measure by dX. Then we have a series of identities related to $E_{G(N)}(n, J)$ which will provide a basis for obtaining tractable expressions for our functions μ_m, and later for ν_j (to be introduced in the near future). Let χ_J be the characteristic function of the interval J. First of all,

$$\sum_{n=0}^{N}(1+z)^n E_{G(N)}(n,J) = \int_{G(N)} \prod_{j=1}^{N}(1+z\chi_J(\theta_j))dX \tag{37}$$

since for any $X \in G(N)$ which has precisely n eigenvalues in J, the integrand is $(1+z)^n$. Expanding out the product on the right side gives

$$\int_{G(N)} \prod_{j=1}^{N}(1+z\chi_J(\theta_j))dX \;=\; \sum_{n=0}^{N} z^n \binom{N}{n} \int_{G(N)} \prod_{j=1}^{n} \chi_J(\theta_j)dX. \tag{38}$$

Next by Gaudin's Lemma, (24)

$$\binom{N}{n} \int_{G(N)} \prod_{j=1}^{n} \chi_J(\theta_j)dX = \frac{z^n}{n!} \int_{J^n} \det_{n\times n} K_{G(N)}(\theta_j,\theta_k)\; d\theta_1 \ldots d\theta_n$$

where $K_{G(N)}(x,y)$ is the appropriate kernel for the group $G(N)$. Thus,

$$\sum_{n=0}^{N} z^n \binom{N}{n} \int_{G(N)} \prod_{j=1}^{n} \chi_J(\theta_j)dX = \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \det_{n\times n} K_{G(N)}(\theta_j,\theta_k)\; d\theta_1 \ldots d\theta_n. \tag{39}$$

Now, for each $G(N)$ we can express

$$K_{G(N)}(x,y) = \sum_{h=1}^{N} \phi_{h,G}(x)\psi_{h,G}(y) \tag{40}$$

for appropriate ϕ and ψ. Therefore, by Gram's identity

$$\begin{aligned} &\sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \det_{n\times n} K_{G(N)}(\theta_j,\theta_k)\; d\theta_1 \ldots d\theta_n \\ &\qquad = \det_{N\times N}\left(I + z\int_J \phi_{j,G}(\theta)\psi_{k,G}(\theta)\; d\theta\right). \end{aligned} \tag{41}$$

Let $M_{J,G(N)}$ denote the $N \times N$ matrix with entries

$$m_{j,k} = \int_J \phi_{j,G}(\theta)\psi_{k,G}(\theta)\; d\theta. \tag{42}$$

Then

$$\det_{N\times N}\left(I + z\int_J \phi_{j,G}(\theta)\psi_{k,G}(\theta)\ d\theta\right) = \prod_{j=1}^{N}\bigl(1 + z\lambda_{j,G(N)}(J)\bigr) \tag{43}$$

where the $\lambda_{j,G(N)}(J)$ are the eigenvalues of $M_{J,G(N)}$.

We claim that if the kernel is symmetric (i.e. $K_{G(N)}(x,y) = K_{G(N)}(y,x)$), then the eigenvalues of $M_{J,G(N)}$ are also the eigenvalues of the integral operator $K_{J,G(N)}$ defined by

$$(K_{J,G(N)}f)(\theta) \;=\; \int_J K_{G(N)}(\theta,\mu)f(\mu)\ d\mu \tag{44}$$

acting on the N-dimensional space generated by $\{\psi_j(x) : 1 \le j \le N\}$.

Proof. Suppose that λ is an eigenvalue of $M_{J,G(N)}$ corresponding to an eigenvector $\vec{v} = (b_1, \ldots, b_N)'$ where the prime indicates transpose. Then, for each j,

$$\lambda b_j = \sum_{k=1}^{N} m_{jk}b_k = \sum_{k=1}^{N} b_k \int_J \phi_j(\theta)\psi_k(\theta)\ d\theta$$

for each j. Multiplying both sides by $\psi_j(\mu)$ and summing over j, we obtain

$$\begin{aligned}
\lambda\sum_{j=1}^{N} b_j\psi_j(\mu) \;&=\; \int_J \bigl(\sum_{j=1}^{N}\phi_j(\theta)\psi_j(\mu)\bigr)\bigl(\sum_{k=1}^{N} b_k\psi_k(\theta)\bigr)\ d\theta \\
&=\; \int_J K_{G(N)}(\theta,\mu)\bigl(\sum_{k=0}^{N} b_k\psi_k(\theta)\bigr)\ d\theta \\
&=\; \int_J K_{G(N)}(\mu,\theta)\bigl(\sum_{k=1}^{N} b_k\psi_k(\theta)\bigr)\ d\theta = K_{J,G(N)}\sum_{k=1}^{N} b_k\psi_k(\mu)
\end{aligned}$$

so that λ is an eigenvalue of $K_{J,G(N)}$ corresponding to the eigenfunction $f(\mu) = \sum_{k=1}^{N} b_k\psi_k(\mu)$.

Recapitulating, we have found that

$$\begin{aligned}
\sum_{n=0}^{N}(1+z)^n E_{G(N)}(n,J) \;&=\; \sum_{n=0}^{N}\frac{z^n}{n!}\int_{J^n}\det_{n\times n} K_{G(N)}(\theta_j,\theta_k)\ d\theta_1\ldots d\theta_n \\
&=\; \prod_{j=1}^{N}\bigl(1 + z\lambda_{j,G(N)}(J)\bigr)
\end{aligned} \tag{45}$$

where the $\lambda_{j,G(N)}(J)$ are the eigenvalues of the integral operator $K_{J,G(N)}$ defined by

$$(K_{J,G(N)}f)(\theta) \;=\; \int_J K_{G(N)}(\theta,\mu)f(\mu)\;d\mu.$$

It can be shown that this equation scales appropriately for each G so that the large N limit can be taken. This results in (with an obvious notation E_G)

$$\begin{aligned} \sum_{n=0}^{\infty}(1+z)^n E_G(n,J) &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{J^n}\det_{n\times n}K_G(\theta_j,\theta_k)\;d\theta_1\ldots d\theta_n \qquad (46)\\ &= \prod_{j=1}^{\infty}\bigl(1+z\lambda_{j,G}(J)\bigr) \end{aligned}$$

where the $\lambda_{j,G}(J)$ are the eigenvalues of the integral operator $K_{J,G}$ defined by

$$(K_{J,G}f)(\theta) \;=\; \int_J K_G(\theta,\mu)f(\mu)\;d\mu.$$

The function $F(z)$ of (35) is equal to each of the above with G=U. Thus, we find for $\mu_m(s)$ that

$$\mu_m(s) = \frac{d^2}{ds^2}\frac{d^{m-1}}{dz^{m-1}}\left(z^{-2}\prod_{j=1}^{\infty}(1+z\lambda_{j,U}([0,s]))\right)\Bigg|_{z=-1}. \qquad (47)$$

9.4 jth lowest eigenvalue

Let

$$\nu_{G(N)}(j,s)$$

be the density function for the jth lowest eigenvalue so that

$$\begin{aligned} &\text{meas}\{A\in G(N):\text{ the } j\text{th eigenvalue } \theta_j \text{ is smaller than } s\} \qquad (48)\\ &\quad= \int_0^s \nu_{G(N)}(j,x)\;dx. \end{aligned}$$

Then the set of $A \in G(N)$ with $\theta_j > s$ is the disjoint union of the set of A with exactly n eigenangles in $[0, s]$ for $n = 0, 1, \dots, j-1$. Thus,

$$\int_s^\infty \nu_{G(N)}(j, x) dx = \sum_{n=0}^{j-1} E_{G(N)}(n, [0, s]).$$

Therefore, by (45), we have

$$\nu_{G(N)}(j, s) = -\frac{d}{ds} \sum_{n=0}^{j-1} \frac{d^n}{dz^n} \prod_{n=1}^{N} (1 + z\lambda_{G(N),n}([0, s]))\Big|_{z=-1}. \tag{49}$$

In the large N limit, this becomes

$$\nu_G(j, s) = -\frac{d}{ds} \sum_{n=0}^{j-1} \frac{d^n}{dz^n} \prod_{n=1}^{\infty} (1 + z\lambda_{G,n}([0, s]))\Big|_{z=-1}. \tag{50}$$

For example,

$$\nu_G(1, s) = -\frac{d}{ds} \prod_{n=1}^{\infty} (1 - \lambda_{G,n}([0, s]). \tag{51}$$

9.5 Relations between the eigenvalues

In this section, we develop a relationship between the eigenvalues λ_U and the eigenvalues λ_O and λ_S. (42). In the case that $J = [-s, s]$, note that if $\psi(\theta)$ is an eigenfunction of $M_{[-s,s],U(N)}$ with eigenvalue λ then $\psi(-\theta)$ is also an eigenfunction with eigenvalue λ, since

$$\lambda\psi(\theta) = \int_{-s}^{s} S_N(\theta - \mu)\psi(\mu)\ d\mu$$

implies that

$$\begin{aligned} \lambda\psi(-\theta) &= \int_{-s}^{s} S_N(-\theta - \mu)\psi(\mu)\ d\mu \\ &= \int_{-s}^{s} S_N(\theta + \mu)\psi(\mu)\ d\mu \\ &= \int_{-s}^{s} S_N(\theta - \mu)\psi(-\mu)\ d\mu. \end{aligned}$$

Therefore, if $\psi(\theta) + \psi(-\theta) \neq 0$, then it is also an eigenfunction with eigenvalue λ. A similar comment holds for $\psi(\theta) - \psi(-\theta)$. Consequently, each eigenfunction can be taken to be even or odd. The even eigenfunctions are also eigenfunctions of the integral equation with kernel

$$\frac{S_N(\mu - \theta) + S_N(\mu + \theta)}{2}$$

and the odd eigenfunctions are also eigenfunctions of the integral equation with kernel

$$\frac{S_N(\mu - \theta) - S_N(\mu + \theta)}{2}.$$

In general, if a matrix b is a "checkerboard" matrix, then the determinant of b factors. Specifically, if $b_{j,k} = 0$ whenever $i + j$ is odd, then

$$\det_{N\times N}(b_{j,k}) = \det_{[(N+1)/2]}(b_{2i-1,2j-1}) \det_{[N/2]\times[N/2]}(b_{2i,2j})$$

where $[x]$ is the greatest integer less than or equal to x.

We have such a factorization for $\det(I - M_{[-s,s],U(N)})$. Using the fact that

$$\sum_h (\delta_{jh} - \cos(j\theta)\cos(h\theta))(\delta_{hk} - \sin(h\theta)\sin(k\theta)) = \delta_{jk} - \cos(k-j)\theta$$

we deduce from (42) (see also Mehta (10.2.6)) that

$$\det(I - M_{[-s,s],U(N)}) = \det(I - M_{[-s,s],SO(2N)}) \det(I - M_{[-s,s],USp(2N)}).$$

This gives a factorization

$$\prod_{n=1}^{2N}(1 - \lambda_{n,U(2N)}(s)) = \prod_{n=1}^{N}(1 - \lambda_{n,SO(2N)}(s))(1 - \lambda_{n,USp(2N)}(s)) \qquad (52)$$

into even and odd eigenvalues. In particular, in the limit we have

$$\prod_{n=1}^{\infty}(1 - \lambda_{n,U}(s)) = \prod_{n=1}^{\infty}(1 - \lambda_{n,Sp}(s))(1 - \lambda_{n,SO,even}(s)). \qquad (53)$$

Alternatively, we have

$$\prod_{n=1}^{\infty}(1 - \lambda_{n,Sp}(s)) = \prod_{n=1}^{\infty}(1 - \lambda_{2n,U}(s)) \qquad (54)$$

and

$$\prod_{n=1}^{\infty}(1-\lambda_{n,SO,even}(s)) = \prod_{n=1}^{\infty}(1-\lambda_{2n-1,U}(s)) \tag{55}$$

provided that the $\lambda_{n,U}(s)$ are indexed so that an even index n corresponds to an even eigenfunction and an odd index n is for an odd eigenfunction of the integral operator (44) with kernel $K_U(x,y) = S(x-y)$. These formulae can be used to give expressions for $\nu_G(j,s)$ in terms of the eigenvalues $\lambda_{n,U}(s)$.

10 Conclusion

These notes have introduced four of the basic eigenvalue statistics for the groups of particular interest to number theorists and have shown the preliminary steps needed to make these statistics somewhat tractable. There are many directions to go after this basic introduction. Topics such as Painlevé equations, Toeplitz operators, the Szego limit theorems, and averages of characteristic polynomials are among the many developments that are essential for a more full introduction to random matrix theory. Many of these are covered elsewhere in this volume.

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