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Bounds for the coefficients of powers of the Δ-function

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Abstract

For $k \geqslant 1$, let $\sum_{n=k}^{\infty} \tau_k(n) q^n = q^k \prod_{n=1}^{\infty} (1-q^n)^{24k}$. It follows from Deligne's proof of the Weil conjectures that there is a constant C_k so that $|\tau_k(n)| \leqslant C_k d(n) n^{(12k-1)/2}$. We study the value of C_k as a function of k, and show that it tends to zero very rapidly.

1. *Introduction and statement of results*

For an integer r, define the numbers $p_r(n)$ by

$$\sum_{n=0}^{\infty} p_r(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^r.$$

For various values of r, these numbers capture important arithmetic objects. For example, when $r = -1$, we recover the classical partition generating function

$$\sum_{n=0}^{\infty} p_{-1}(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

while for $r = 1$ and $r = 3$ we recover the identities of Euler and Jacobi,

$$\sum_{n=0}^{\infty} p_1(n) q^n = \prod_{n=1}^{\infty} (1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2},$$

$$\sum_{n=0}^{\infty} p_3(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{(n^2+n)/2}.$$

In a series of papers ([**9**–**12**]), Newman studied the function $p_r(n)$, and proved a number of identities for it. Newman was particularly interested in when the function $p_r(n)$ is zero and computed $p_r(n)$ for small n (as a polynomial in r). These coefficients were later considered by many authors, including Gupta, Atkin, Costello, Gordon, and finally Serre. Serre [**16**] showed that if r is an even integer, then $\{n : p_r(n) = 0\}$ has density zero if and only if $r = 2, 4, 6, 8, 10, 14$, or 26.

Another natural question is about how large (as a function of r and n) the coefficients $p_r(n)$ are. In this regard, Newman's approach of expressing the coefficients $p_r(n)$ as polynomials in r is very ineffective. A stronger result follows from the work of Deligne [**2**] (at least when r is even) and gives that $p_r(n) \ll n^{(r-1)/2+\epsilon}$. In the case of $r = 24$, it implies Ramanujan's famous conjecture that if

$$\sum_{n=1}^{\infty} p_{24}(n-1) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

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is the Fourier expansion of the weight 12 cusp form $\Delta(z)$, then

$$|p_{24}(n-1)| \leqslant d(n)n^{11/2},$$

where $d(n)$ is the number of divisors of n.

Deligne's bound applies to cuspidal Hecke eigenforms of all weights. Hence, if $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_m$ is any cusp form of weight m, then by writing $f(z) = \sum_{i=1}^{\dim S_m} c_i f_i$, where the f_i are normalized Hecke eigenforms, we have that $|a(n)| \leqslant Cd(n)n^{(m-1)/2}$, where $C = \sum_{i=1}^{\dim S_m} |c_i|$.

For example, we may write

$$\Delta^2(z) = \sum_{n=2}^{\infty} p_{48}(n-2)q^n = q^2 - 48q^3 + 1080q^4 + \ldots \in S_{24}$$

as a linear combination of the Hecke eigenforms

$$f_1(z) = q + (540 + 12\sqrt{144169})q^2 + (169740 - 576\sqrt{144169})q^3 + \ldots,$$
$$f_2(z) = q + (540 - 12\sqrt{144169})q^2 + (169740 + 576\sqrt{144169})q^3 + \ldots.$$

Then, we have that

$$\Delta^2(z) = \frac{f_1 - f_2}{24\sqrt{144169}},$$

and hence $|p_{48}(n-2)| \leqslant (1/12\sqrt{144169})d(n)n^{23/2}$. Note that $(1/12\sqrt{144169}) \approx 0.000219$ is quite small.

The aim of this paper is to compute explicit bounds for the coefficients $p_r(n)$, when $r \geqslant 0$ and is a multiple of 24. We then have that

$$\Delta^k(z) := \sum_{n=k}^{\infty} p_{24k}(n-k)q^n.$$

Let $C_k := \sum_{i=1}^{k} |c_i|$, where $\Delta^k(z) = \sum_{i=1}^{k} c_i f_i$ is the representation of Δ^k as a sum of Hecke eigenforms. Then

$$|p_{24k}(n-k)| \leqslant C_k d(n) n^{(12k-1)/2}.$$

It suffices therefore to bound C_k. Our main result is the following theorem.

THEOREM 1. *For $k \geqslant 2$, we have*

$$\log(C_k) = -6k\log(k) + 6k\log\left(\frac{2\pi^3 e}{27\Gamma(2/3)^6}\right) + O(\log(k)).$$

This result follows from explicit upper and lower bounds on C_k derived below. Our approach is as follows. For $f, g \in S_k$, let

$$\langle f, g\rangle_k = \frac{3}{\pi}\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(x+iy)\overline{g(x+iy)}y^k \,\frac{dx\,dy}{y^2}$$

be the normalized Petersson inner product of f and g. Elementary considerations provide bounds on $\langle\Delta^k, \Delta^k\rangle_{12k}$. If $f_i \in S_{12k}$ is a normalized Hecke eigenform, then $\langle f_i, f_i\rangle_{12k}$ is essentially the special value at $s = 1$ of the symmetric square L-function associated to f_i. Goldfeld, Hoffstein, and Lieman showed in the appendix to [**5**], that such an L-function can have no Siegel zero. We make their argument explicit and derive an explicit lower bound on $\langle f_i, f_i\rangle_{12k}$.

These bounds are translated to bounds on C_k using the well-known fact (see [**6**, Theorem 6.12]) that if $f_i \neq f_j$ are Hecke eigenforms, then $\langle f_i, f_j\rangle_{12k} = 0$.

Remark. It is plausible that in fact

$$C_k = \sup_{n\geqslant 1} \frac{|p_{24k}(n-k)|}{d(n)n^{(12k-1)/2}}.$$

This would follow if for each eigenform $f_i = \sum_{n=1}^{\infty} a_i(n)q^n$, we have $|a_i(p)| \geqslant (2-\epsilon)p^{(12k-1)/2}$ for a positive density set of primes, and if the coefficients $a_1(p), a_2(p), \ldots, a_k(p)$ are 'independent'. The first statement would follow from the Sato–Tate conjecture. Recently, Richard Taylor has achieved an important breakthrough by proving the Sato–Tate conjecture for a wide class of elliptic curves. Taylor's work establishes the automorphy of symmetric power L-functions, which can be used (as in [**14**]) to produce lower bounds for Hecke eigenvalues.

Remark. The approach given here readily generalizes to powers of any fixed modular form, provided that the powers are orthogonal to CM forms. One cannot (at present) exclude the possible existence of a Siegel zero for the symmetric square of a CM form. For $r \equiv 0, 12, 16 \pmod{24}$, we can relate $\sum p_r(n)q^n$ to a modular form lying in a space with no CM forms.

In Section 2 we derive upper and lower bounds on the Petersson norms $\langle \Delta^k, \Delta^k \rangle_{12k}$ and $\langle f_i, f_i \rangle_{12k}$. In Section 3 we use the results derived in Section 2 to prove Theorem 1, and in the Appendix we present some numerical data.

2. *Petersson norm bounds*

First, we will compute bounds for the Petersson norm of Hecke eigenforms $f_i \in S_{12k}$. We will repeatedly use the fact (see the second equation [**6**, p. 251]) that

$$L(\mathrm{Sym}^2 f_i, 1) = \frac{6}{\pi^2} \cdot \frac{(4\pi)^{12k} \langle f_i, f_i \rangle_{12k}}{\Gamma(12k)}.$$

If the normalized L-function of $f_i = \sum_{n=1}^{\infty} a_i(n)q^n$ is

$$L(f_i, s) = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1},$$

where $\alpha_p + \beta_p = a_i(p)/p^{(12k-1)/2}$ and $\alpha_p\beta_p = 1$, then

$$L(\mathrm{Sym}^2 f_i, s) = \prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1}.$$

This L-function is known by work of Gelbart and Jacquet [**3**] to be the L-function of a cuspidal automorphic representation on GL(3). Hence, it is entire and if

$$\Lambda(\mathrm{Sym}^2 f_i, s) = \pi^{-3s/2}\Gamma((s+1)/2)\Gamma((s+(12k-1))/2)\Gamma((s+12k)/2)L(\mathrm{Sym}^2 f_i, s),$$

then $\Lambda(\mathrm{Sym}^2 f_i, s) = \Lambda(\mathrm{Sym}^2 f_i, 1-s)$.

Lemma 2. *If $f_i \in S_{12k}$ is a normalized Hecke eigenform, then*

$$L(\mathrm{Sym}^2 f_i, s) \neq 0$$

for $s > 1 - (5-2\sqrt{6})/10\log(12k)$.

Proof. Goldfeld, Hoffstein, and Lieman introduce the auxiliary function

$$L(s) = \zeta(s)^2 L(\mathrm{Sym}^2 f_i, s)^3 L(\mathrm{Sym}^4 f_i, s).$$

Here,

$$L(\mathrm{Sym}^4 f_i, s) = \prod_p (1-\alpha_p^4 p^{-s})^{-1}(1-\alpha_p^2 p^{-s})^{-1}(1-p^{-s})^{-1}(1-\alpha_p^{-2}p^{-s})^{-1}(1-\alpha_p^{-4}p^{-s})^{-1}.$$

The work of Kim [7] implies that this is the L-function of a cuspidal automorphic representation on $\mathrm{GL}(5)$. From this, it follows that $L(\mathrm{Sym}^4 f_i, s)$ has an analytic continuation and functional equation of the usual type (see the paper of Cogdell and Michel [1] for details about computing the sign of the functional equation and the Γ-factors of symmetric power L-functions using the local Langlands correspondence for $\mathrm{GL}(n)$).

If we let $\Lambda(s) = s^2(1-s)^2 G(s) L(s)$, where

$$\begin{aligned} G(s) = \pi^{-16s/2}&\Gamma(s/2)^3\Gamma((s+1)/2)^3\Gamma((s+(12k-1))/2)^4\Gamma((s+12k)/2)^4 \\ &\Gamma((s+(24k-2))/2)\Gamma((s+(24k-1))/2), \end{aligned}$$

then $\Lambda(1-s) = \Lambda(s)$. Writing $\Lambda(s) = e^{A+Bs}\prod_\rho (1-s/\rho)\, e^{s/\rho}$ and taking the logarithmic derivative gives

$$\sum_\rho \frac{1}{s-\rho} + \frac{1}{\rho} = \frac{2}{s} + \frac{2}{1-s} + \frac{L'(s)}{L(s)} + \frac{G'(s)}{G(s)} - B.$$

Now, the Dirichlet coefficients of $L(s)$ are non-negative. This implies that for $\mathrm{Re}(s) > 1$, we get $L'(s)/L(s) < 0$. Taking the real part of this equation and noting that $\mathrm{Re}(B) = -\sum_\rho \mathrm{Re}(1/\rho)$ gives

$$\sum_\rho \mathrm{Re}\left(\frac{1}{s-\rho}\right) \leqslant \frac{2}{s} + \frac{2}{1-s} + \frac{G'(s)}{G(s)}.$$

Assume that $s = 1+\alpha$, where $0 < \alpha \leqslant 1/2$ will be chosen later. Noting that $\Gamma'(s)/\Gamma(s) \leqslant \log(s)$ for $s \geqslant 1$ gives that, in this range, $G'(s)/G(s) \leqslant 10\log(12k) - 2$.

Suppose that $L(\mathrm{Sym}^2 f, \beta) = 0$. Then we have

$$\frac{3}{\alpha + 1 - \beta} \leqslant \frac{2}{\alpha} + 10\log(12k).$$

Solving for β and choosing α optimally yields the desired result. □

Next, we follow the argument of Hoffstein [4] to translate this into an explicit lower bound on $L(\mathrm{Sym}^2 f_i, 1)$.

Lemma 3. *If $f \in S_{12k}$ is a normalized Hecke eigenform, then*

$$L(\mathrm{Sym}^2 f, 1) > \frac{1}{64\log(12k)}.$$

Proof. Let

$$L(f\otimes f, s) = \zeta(s) L(\mathrm{Sym}^2 f, s) = \sum_{n=1}^\infty \frac{a(n)}{n^s}.$$

Then $a(n) \geqslant 0$ for all $n \geqslant 1$. Also, its functional equation is well known (for example, it follows from that of $L(\mathrm{Sym}^2 f, s)$).

Let $\beta = 1 - (5-2\sqrt{6})/10\log(12k)$. We set $x = (12k)^A$. It will turn out that the optimal A is about $8/5$ and we choose $A = 8/5 + 10/\log(12k)$. We consider

$$I = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L(f\otimes f, s+\beta)x^s\, ds}{s\prod_{k=2}^{10}(s+k)}.$$

We use the fact that

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{x^s\,ds}{s\prod_{k=2}^{10}(s+k)}=\begin{cases}\dfrac{(x+9)(x-1)^9}{10!x^{10}}, & x>1\\ 0, & x<1,\end{cases}$$

and conclude that

$$I=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{L(f\otimes f,s+\beta)x^s}{s\prod_{k=2}^{10}(s+k)}=\sum_{n\leqslant x}\frac{a(n)\,(x/n+9)\,(x/n-1)^9}{10!n^{\beta}(x/n)^{10}}.$$

One can easily show that $a(n^2)\geqslant 1$. We consider only those terms for which $x/n\geqslant 559$. This gives a lower bound on the integral of $1.6442234/10!$.

Now, we move the contour to $\mathrm{Re}(s)=\alpha$, where $\alpha=-3/2-\beta$. The poles at $s=1-\beta$, $s=0$ and $s=-2$ contributed residues. This gives

$$\begin{aligned}I=&\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{L(f\otimes f,s+\beta)x^s\,ds}{s\prod_{k=2}^{10}(s+k)}+\frac{L(\mathrm{Sym}^2 f,1)x^{1-\beta}}{(1-\beta)\prod_{k=2}^{10}(1-\beta+k)}\\&+\frac{L(f\otimes f,\beta)}{10!}+\frac{L(f\otimes f,-2+\beta)x^{-2}}{2\cdot 8!}.\end{aligned}$$

There are no zeroes of $L(\mathrm{Sym}^2 f,s)$ to the right of β and hence $L(\mathrm{Sym}^2 f,\beta)\geqslant 0$. Since $\zeta(\beta)<0$, it follows that $L(f\otimes f,\beta)\leqslant 0$. Also $L(f\otimes f,-2+\beta)<0$. It follows that

$$\frac{1.6442234}{10!}-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{L(f\otimes f,s+\beta)x^s\,ds}{s\prod_{k=2}^{10}(s+k)}\leqslant\frac{L(\mathrm{Sym}^2 f,1)x^{1-\beta}}{(1-\beta)\prod_{k=2}^{10}(1-\beta+k)}.\tag{1}$$

Now, we bound the integral in the above inequality. The functional equation for $L(f\otimes f,s)$ implies that

$$\frac{|L(f\otimes f,-3/2+it)|}{|L(f\otimes f,5/2-it)|}=|1/2+it|^2|3/2+it|^2\prod_{m=1}^{4}|12k-3+m/2+it|.$$

Also $|L(f\otimes f,5/2-it)|\leqslant\zeta(5/2)^4$. Hence $|I|$ is bounded above by

$$\begin{aligned}&\frac{\zeta(5/2)^4}{2^9\pi^9}(12k)^{A(-3/2-\beta)}\cdot\int_{-\infty}^{\infty}\frac{|1/2+it|^2|3/2+it|^2\prod_{m=1}^{4}|12k-3+m/2+it|\,dt}{|9/4+it||1/4+it|\prod_{n=3}^{10}|n-5/2+it|}\\&\leqslant\frac{\zeta(5/2)^4(12k)^{4-A(3/2+\beta)}}{2^9\pi^9}\int_{-\infty}^{\infty}\frac{|1/2+it||3/2+it||1+it|^3|25/24+it|}{|1/4+it||9/4+it|\prod_{n=2}^{7}|n+1/2+it|}\\&\leqslant\frac{(12k)^{4-A(3/2+\beta)}\cdot 0.181266}{10!}.\end{aligned}$$

Hence, returning to equation (1), we have

$$L(\mathrm{Sym}^2 f,1)\geqslant(1-\beta)\left(\frac{1.6442234}{(12k)^{A(1-\beta)}}-\frac{0.181266}{(12k)^{(5/2)A-4}}\right).$$

We choose $A=8/5+10/\log(12k)$ and obtain the desired result. □

Next, we use an elementary argument to obtain an upper bound for $\langle f_i,f_i\rangle_{12k}$.

Lemma 4. *If f_i is a normalized Hecke eigenform of weight k and $k\geqslant 48$, then*

$$\langle f_i,f_i\rangle_k\leqslant 3.182\frac{\Gamma(k)\log^3(k)}{(4\pi)^k}.$$

Remark. This result could also be obtained from the convexity bound for $L(\mathrm{Sym}^2 f_i,s)$.

Proof of Lemma 4. For brevity we will explain only the main ideas. One can extend the integral for $\langle f_i, f_i\rangle_k$ to the region $\{x+iy : -1/2 \leqslant x \leqslant 1/2, y \geqslant \sqrt{3}/2\}$. If

$$f_i(z) = \sum_{n=1}^{\infty} a_i(n) q^n,$$

and we replace $f_i(z)$ by its Fourier expansion, then we obtain the upper bound

$$\langle f_i, f_i\rangle_k \leqslant \frac{3}{\pi} \sum_{n=1}^{\infty} |a_i(n)|^2 \int_{\sqrt{3}/2}^{\infty} e^{-4\pi n y} y^{k-2}\, dy.$$

Changing variables, we get

$$\frac{1}{(4\pi)^{k-1}} \sum_{n=1}^{\infty} \frac{|a_i(n)|^2}{n^{k-1}} \int_{\pi\sqrt{3}n}^{\infty} e^{-u} u^{k-2}\, du.$$

The Deligne bound implies that $|a_i(n)|^2/n^{k-1} \leqslant d(n)^2$. The integrand also does not depend on n. Replacing the order of the sum and the integral we obtain

$$\frac{1}{(4\pi)^{k-1}} \int_{\pi\sqrt{3}}^{\infty} u^{k-2} e^{-u} \sum_{n\leqslant u} d(n)^2\, du.$$

An asymptotic for $\sum_{n\leqslant u} d(n)^2$ was given by Ramanujan [**13**, Equation (B)]. The elementary proof of $\sum_{n\leqslant u} d(n)^2 \sim (1/\pi^2) u \log^3(u)$ in [**8**, Theorem 7.8], can be easily modified to show that $\sum_{n\leqslant u} d(n)^2 \leqslant (19/3 \log^3(6)) u \log^3(u)$ for all $u \geqslant 1$. Hence, it suffices to estimate

$$\int_{\pi\sqrt{3}}^{\infty} e^{-u} u^{k-2} \log^3(u)\, du.$$

One can easily check that the integrand decays rapidly for $u \gg k\log(k)$. The remainder is easy to estimate by comparison with the Γ-function. $\square$

The next result is of independent interest and is useful in bounding $\langle \Delta^k, \Delta^k\rangle_{12k}$.

Lemma 5. *Let $f(x,y) = |\Delta(x+iy)|^2 y^{12}$. Then for $y > 0$ we have*

$$f(x,y) \leqslant B := \left(\frac{\sqrt{2}\pi}{3\Gamma(2/3)^3}\right)^{24}$$

with equality if and only if $x+iy = (a\omega+b)/(c\omega+d)$ for $a,b,c,d \in \mathbb{Z}$ with $ad-bc=1$ and $\omega = (-1+i\sqrt{3})/2$.

Proof. First, the equality when $x = -1/2$ and $y = \sqrt{3}/2$ is very classical (see, for example, [**15**, equation (2), p. 110]).

Next, the function $|\Delta(z)|^2 \mathrm{Im}(z)^{12}$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. It suffices therefore to find its maximum on the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, namely $\{z \in \mathbb{H} : -1/2 \leqslant \mathrm{Re}(z) \leqslant 1/2 \text{ and } |z| \geqslant 1\}$. Moreover, since the Fourier coefficients of $\Delta(z)$ are real, it follows that $\overline{\Delta(x+iy)} = \Delta(-x+iy)$. Thus $f(x,y) = f(1-x,y)$, and it suffices to consider $-1/2 \leqslant x \leqslant 0$.

We approximate the size of $|\Delta(x+iy)|$ by $\left|\sum_{n=1}^{4} \tau(n) q^n\right|$. We can easily see that for any y this is maximized when $x = -1/2$. One can also show that $y^6 \left|\sum_{n=1}^{4} \tau(n) q^n\right|$ is maximized when $y = \sqrt{3}/2$. It follows from this that if $f(x,y) \geqslant f(-1/2, \sqrt{3}/2)$ for $x+iy$ in the fundamental domain, then $y \leqslant 0.8676$ and hence $-1/2 \leqslant x \leqslant -0.497$.

Differentiating the equality $f(x,y) = f(1-x,y)$ with respect to x and setting $x = -1/2$ shows that $f_x(-1/2,y) = 0$ for all y. Using the transformation law with the matrix $\left[\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ shows that

$$f\left(\frac{-x^2-y^2-x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) = f(x,y).$$

Differentiating this with respect to x, setting $x = -1/2$, $y = \sqrt{3}/2$ and using that $f_x(-1/2,\sqrt{3}/2) = 0$ shows that $f_y(-1/2,\sqrt{3}/2) = 0$. Since the maximum of $f(x,y)$ occurs where f_x and f_y both vanish, it suffices to show that this does not occur elsewhere in the box $-1/2 \leqslant x \leqslant -0.497$, $\sqrt{3}/2 \leqslant y \leqslant 0.8676$.

Next, we use the product expansion $f(x,y) = y^{12}\prod_{n=1}^{\infty}|1-q^n|^{48}$. This implies that

$$\frac{f_x}{f} = 24\sum_{n=1}^{\infty}\frac{2\pi n\sin(2\pi nx)e^{-2\pi ny}}{1-2\cos(2\pi nx)e^{-2\pi ny}+e^{-4\pi ny}}.$$

We note that

$$\frac{f_{xx}}{f} = \frac{d}{dx}\left(\frac{f_x}{f}\right) + \left(\frac{f_x}{f}\right)^2.$$

Trivially estimating f_x/f, we see that $|f_x/f| \leqslant 0.665$ in this box. We estimate all but the first two terms of $\frac{d}{dx}(f_x/f)$ trivially and obtain the bound $f_{xx}/f \leqslant -1.9$.

Now, we assume that $x = -1/2$. Using

$$\frac{f_y}{f} = \frac{12}{y} - 4\pi + 96\pi\sum_{n=1}^{\infty}\frac{(-1)^n ne^{-2\pi ny}}{1-(-1)^n e^{-2\pi ny}},$$

we will estimate

$$\frac{f_{yy}}{f} = \frac{d}{dy}\left(\frac{f_y}{f}\right) + \left(\frac{f_y}{f}\right)^2.$$

We see that $|f_y/f| \leqslant 0.048$. The main term is $-12/y^2$, and for $y \leqslant 1.1$, this dominates and $f_{yy}/f < 0$. This establishes the desired result since we have $f_x < 0$ for $x \neq -1/2$, and if $x = -1/2$ then we have that $f_x = 0$ and $f_y < 0$ unless $y = \sqrt{3}/2$. □

With a little bit of work, the above lemma can be translated into bounds on $\langle\Delta^k, \Delta^k\rangle_{12k}$.

Lemma 6. *For $k \geqslant 1$, we have*

$$\frac{0.08906B^k}{k} \leqslant \langle\Delta^k,\Delta^k\rangle_{12k} \leqslant \frac{76.4B^k}{k}.$$

Proof. For the lower bound, similar arguments to those in the proof of Lemma 5 imply that for all x and y, $f_{xx} \geqslant -4.251\cdot f$ and for $x = -1/2$, $f_{yy} \geqslant -8.652\cdot f$. Using the upper bound on f established above, we obtain that if $C := -3.555\cdot 10^{-5}$, then $f_{xx} \geqslant C$ for all x and y and $f_{yy} \geqslant C$ for $x = -1/2$ and $y \geqslant \sqrt{3}/2$. Integrating from $(-1/2,\sqrt{3}/2)$ to $(-1/2,y)$ and then to (x,y) shows that

$$f(x,y) - f(-1/2,\sqrt{3}/2) \leqslant -(C/2)((x+1/2)^2 + (y-\sqrt{3}/2)^2).$$

If we fix $\epsilon > 0$ then it follows that $f(x,y) \geqslant B - \epsilon$ on a set of measure at least $(2\pi/3(3.555\cdot 10^{-5}))\epsilon$. This gives a lower bound for the Petersson norm of

$$\frac{2\pi}{3(3.555\cdot 10^{-5})}\epsilon(B-\epsilon)^k.$$

This is maximized with $\epsilon = B/(k+1)$ and gives the desired result.

For the upper bound, we let

$$S_\epsilon = \{(x,y) : -1/2 \leqslant x \leqslant 1/2, x^2+y^2 \geqslant 1, \text{ and } f(x,y) \leqslant B-\epsilon\}.$$

One can check that there is a constant C_2 so that if ϵ is small enough then $\mu(S_\epsilon) \leqslant C_2\epsilon$, where $\mu = (3/\pi)(dx\,dy/y^2)$. Choose ϵ small enough so that $\mu(S_\delta) \leqslant C_2\delta$ for all $\delta \leqslant \epsilon$ and let n be a positive integer. For $(x,y) \in S_{(l+1)\epsilon/n} - S_{l\epsilon/n}$, we have $f(x,y) \leqslant B - \epsilon l/n$. It follows that

$$\langle \Delta^k, \Delta^k \rangle_{12k} \leqslant \sum_{l=0}^{n-1} \mu\left(S_{(l+1)\epsilon/n} - S_{l\epsilon/n}\right)\left(B - \frac{\epsilon l}{n}\right)^k + \frac{3}{\pi}(B-\epsilon)^k.$$

Let $a_l = \mu\left(S_{(l+1)\epsilon/n} - S_{l\epsilon/n}\right)$ and $b_l = (B-\epsilon l/n)^k$. Notice that $b_0 \geqslant b_1 \geqslant \ldots \geqslant b_{n-1}$. Also for $0 \leqslant m \leqslant n-1$,

$$\sum_{l=0}^{m} a_l = \mu\left(S_{(m+1)\epsilon/n}\right) \leqslant C_2(m+1)\epsilon/n.$$

Since the b_l are decreasing, the sum $\sum_{l=0}^{n-1} a_l b_l \leqslant \sum_{l=0}^{n-1}(C_2\epsilon/n)b_l$. It follows that

$$\langle \Delta^k, \Delta^k \rangle_{12k} \leqslant C_2 \sum_{l=0}^{n-1} \frac{\epsilon}{n}\left(B - \frac{\epsilon l}{n}\right)^k + \frac{3}{\pi}(B-\epsilon)^k.$$

Taking the limit as $n \to \infty$ gives that the first term above is

$$\int_0^\epsilon C_2(B-x)^k\,dx = \frac{C_2}{k+1}\left[B^{k+1} - (B-\epsilon)^{k+1}\right].$$

Hence we have

$$\langle \Delta^k, \Delta^k \rangle_{12k} \leqslant \frac{C_2 B^{k+1}}{k+1}\left[1 - (1-\epsilon/B)^{k+1} + \frac{3(k+1)}{\pi B C_2}(1-\epsilon/B)^k\right].$$

Computations similar to those above show that we may take $\epsilon = 1.93553 \times 10^{-8}$, and $C_2 = 729582$. This gives that for $k \geqslant 1$,

$$\langle \Delta^k, \Delta^k \rangle_{12k} \leqslant \frac{76.4B^k}{k}.$$

Note however, that for small k and for $k \geqslant 300$, the above inequality is better. □

3. *Proof of Theorem 1*

Proof of Theorem 1. We assume that $k \geqslant 4$ and write

$$\Delta^k = \sum_{i=1}^{k} c_i f_i,$$

where the f_i are normalized Hecke eigenforms. Since the Fourier coefficients of the f_i are real, the c_i are real. As noted in the introduction, if $i \neq j$, then $\langle f_i, f_j \rangle = 0$. Computing the inner product of Δ^k with itself, we obtain

$$\langle \Delta^k, \Delta^k \rangle_{12k} = \sum_{i=1}^{k} c_i^2 \langle f_i, f_i \rangle_{12k}.$$

Let B_1 and B_2 be the lower and upper bounds on $\langle f_i, f_i \rangle_{12k}$ furnished by Lemmas 3 and 4, respectively. We obtain

$$\frac{\langle \Delta^k, \Delta^k \rangle_{12k}}{B_2} \leqslant \sum_{i=1}^{k} c_i^2 \leqslant \frac{\langle \Delta^k, \Delta^k \rangle_{12k}}{B_1}.$$

We use Lemma 6 together with the simple inequalities

$$\sqrt{\sum_{i=1}^{k} c_i^2} \leqslant \sum_{i=1}^{k} |c_i| \leqslant \sqrt{k}\sqrt{\sum_{i=1}^{k} c_i^2}$$

to complete the proof. This gives the explicit bound

$$\frac{(4\pi)^{6k} B^{k/2}}{6\sqrt{(12k-1)!}\sqrt{k}\log^{3/2}(12k)} \leqslant C_k \leqslant \frac{55(4\pi)^{6k} B^{k/2} \log^{1/2}(12k)}{\sqrt{(12k-1)!}}$$

Taking logarithms easily yields the desired result. □

Appendix. *Numerical data*

Using MAGMA, if k is small, then we can compute the Fourier expansions of the normalized Hecke eigenforms f_i and hence compute $C_k = \sum_{i=1}^{k} |c_i|$. Table A.1 is a list of k values and the logarithms of the bounds derived in this paper.

TABLE A.1. *Upper bounds, lower bounds and values of* C_k.

k	log(lower bound)	log(C_k)	log(upper bound)
1	−2.9232	0.0000	2.7527
2	−11.706	−8.4243	−4.8448
3	−23.369	−19.657	−15.862
4	−36.977	−33.072	−29.028
5	−52.053	−47.874	−43.769
6	−68.308	−64.102	−59.754
7	−85.549	−81.120	−76.771
8	−103.64	−99.160	−94.665
9	−122.46	−117.84	−113.33
10	−141.96	−137.40	−132.66

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References

1. J. COGDELL and P. MICHEL, 'On the complex moments of symmetric power L-functions at $s = 1$', *Int. Math. Res. Not.* (2004) 1561–1617.
2. P. DELIGNE, 'La conjecture de Weil I', *Publ. Math. Inst. Hautes Études Sci.* (1974) 273–307.
3. S. GELBART and H. JACQUET, 'A relation between automorphic representations of GL(2) and GL(3)', *Ann. Sci. École Norm. Sup.* (4) 11 (1978) 471–542.
4. J. HOFFSTEIN, 'On the Siegel–Tatuzawa theorem', *Acta Arith.* 38 (1980/1981) 167–174.
5. J. HOFFSTEIN and P. LOCKHART, 'Coefficients of Maass forms and the Siegel zero', with an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman, *Ann. of Math.* (2) 140 (1994) 161–181.
6. H. IWANIEC, *Topics in classical automorphic forms*, Graduate Studies in Mathematics 17 (American Mathematical Society, Providence, RI, 1997).
7. H. H. KIM, 'Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2, *J. Amer. Math. Soc.* 16 (2003) 139–183 (electronic).
8. M. B. NATHANSON, *Elementary methods in number theory*, Graduate Texts in Mathematics 195 (Springer, New York, 2000).
9. M. NEWMAN, 'The coefficients of certain infinite products', *Proc. Amer. Math. Soc.* 4 (1953) 435–439.
10. M. NEWMAN, 'An identity for the coefficients of certain modular forms', *J. London Math. Soc.* 30 (1955) 488–493.

11. M. Newman, 'A table of the coefficients of the powers of $\eta(\tau)$', *Nederl. Akad. Wetensch. Proc. Ser. A.* 59 = *Indag. Math.* (N.S.) 18 (1956) 204–216.
12. M. Newman, 'Some theorems about $p_r(n)$', *Canad. J. Math.* 9 (1957) 68–70.
13. S. Ramanujan, 'Some formulæ in the analytic theory of numbers [*Messenger Math.* 45 (1916) 81–84]', *Collected papers of Srinivasa Ramanujan* (AMS Chelsea, Providence, RI, 2000) 133–135.
14. J. Rouse, 'Atkin–Serre type conjectures for automorphic representations on GL(2)', *Math. Res. Lett.* 14 (2007) 189–204.
15. A. Selberg and S. Chowla, 'On Epstein's zeta-function', *J. Reine Angew. Math.* 227 (1967) 86–110.
16. J.-P. Serre, 'Sur la lacunarité des puissances de η', *Glasgow Math. J.* 27 (1985) 203–221.

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