Dirichlet Series for Weighted Convolutions of von Mangoldt Function

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Let $p(x_1, \ldots, x_r)$ be a polynomial with complex coefficients and let

$$\zeta_r(s; p) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} p(m_1, \ldots, m_r)^{-s}, \quad s = \sigma + it.$$

In 1904 Barnes and Mellin proved that $\zeta_4(s; p)$ has a meromorphic continuation to $\mathbb{C}$.

Then in 1995 Essouabri showed that

$$\zeta_r(s_1, \ldots, s_n; p_1, \ldots, p_n) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} p_1(m_1, \ldots, m_r)^{-s_1} \ldots p_n(m_1, \ldots, m_r)^{-s_n}$$

has meromorphic continuation to $\mathbb{C}^n$. A special case of this is the Euler-Zagier r-fold sum

$$\zeta_{EZ,r}(s_1, \ldots, s_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} m_1^{-s_1}(m_1 + m_2)^{-s_2} \ldots (m_1 + \cdots + m_r)^{-s_r}.$$

Let

$$\psi_k(s) = \sum_{m=1}^{\infty} \frac{a_k(m)}{m^s}, \quad \text{for } 1 \le k \le r.$$

Suppose that $\psi_k(s)$

1. is absolutely convergent for $\Re s > \alpha_k > 0$,
2. can be continued meromorphically to $\mathbb{C}$,
3. is holomorphic but for a possible pole of order ≤ 1 at $s = \alpha_k$,
4. is of polynomial order in any fixed strip $\sigma_1 \le \sigma \le \sigma_2$.

In 2003 Matsumoto and Tanigawa showed that the r-fold sum

$$\Psi_r(s_1, \ldots, s_r \mid \psi_1, \ldots, \psi_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{a_1(m_1)}{m_1^{s_1}} \cdots \frac{a_2(m_2)}{(m_1+m_2)^{s_2}} \cdots \frac{a_r(m_r)}{(m_1+\cdots+m_r)^{s_r}}.$$

has a meromorphic continuation to $\mathbb{C}^r$.

Let

$$M(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where $\Lambda(n)$ be the von Mangoldt function. Also, define

$$\phi_2(s) = \Psi_2(0, s; M, M) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Lambda(k)\Lambda(m)}{(k+m)^s} = \sum_{n=1}^{\infty} \frac{G_2(n)}{n^s},$$

where

$$G_2(n) = \sum_{k+m=n} \Lambda(k)\Lambda(n).$$

In 1991 Fuji showed that

$$\sum_{n \le x} G_2(n) = \frac{1}{2}x^2 - 2\sum_{\rho} \frac{x^{1+\rho}}{\rho(1+\rho)} + O((x \log x)^{4/3}), \qquad (1)$$

where ρ runs over the nontrivial zeros of the zeta-function, $\zeta(s)$.

By Perron's formula,

$$\sum_{n\le x} G_2(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \phi_2(s)\frac{x^s}{s}ds + O(T^{-1}x^{2+\epsilon}), \quad c>2.$$

Shifting the path of integration to $\Re(s) = 1+\epsilon$, we see that $x^2/2 - H(x)$ equals the sum of the residues. So $H(x)$ is closely related $\phi_2(s)$ near $\Re(s) = 3/2$.

Also, for even values of n it is expected that $G_2(n) \approx nS_2(n)$, where

$$S_2(n) = \prod_{p|n}\left(1+\frac{1}{p-1}\right)\prod_{(p,n)=1}\left(1+\frac{1}{(p-1)^2}\right),$$

and since Montgomery and Vaughan showed that

$$\sum_{n\le x} nS_2(n) = \frac{1}{2}x^2 + O(x\log x),$$

Fujii's formula in (1) may be rewritten as

$$\sum_{n\le x}(G_2(n) - nS_2(n)) = -H(x) + O((x\log x)^{4/3}).$$

Then in 2007 Egami and Matsumoto proved that $\phi_2(s)$

1. can be continued meromorphically to $\Re s > 1$,
2. is holomorphic except for simple poles at $s = 2$ with residue 1, and at $s = 1 + \rho$ with residue $-2\eta(\rho)/\rho$.

They also introduced the following

Hypothesis (B): Let $\mathcal{I}$ be the set of ordinates of the nontrivial zeros of $\zeta(s)$. There exists a constant α, with $0 < \alpha < \pi/2$, such that if

1. $\gamma_j \in \mathcal{I}$, where $1 \leq j \leq 4$,
2. $\gamma_1 + \gamma_2 \neq 0$,
3. $(\gamma_3, \gamma_4) \neq (\gamma_1, \gamma_2)$ and $(\gamma_3, \gamma_4) \neq (\gamma_2, \gamma_1)$,

then

$$|(\gamma_1 + \gamma_2) - (\gamma_3 + \gamma_4)| \geq \exp(-\alpha(|\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4|)).$$

Egami and Matsumoto showed that, on the Riemann Hypothesis and Hypothesis (B), $\Re s = 1$ is the natural boundary of $\phi_2(s)$. They also conjectured that the error term in Fuji's formula (1) is at most $O(x^{1+\epsilon})$ and $\Omega(x)$.

Now, let $H : \mathbb{R} \to \mathbb{C}$ be periodic with period 1 such that

$$H(t) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi int},$$

and suppose that $H(t)$ is $\mathcal{C}^2$. Also, let $\alpha > 0$ and define

$$F_{2,H,\alpha}(s) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{\Lambda(m_1)\Lambda(m_2)}{(m_1+m_2)^s} H(\alpha \log(m_1+m_2)).$$

Theorem 1. The function $F_{2,H,\alpha}(s)$ is meromorphic in $\Re s > 3/2$, with poles at $s = 2 + 2\pi i \alpha n$, where $n \in \mathbb{Z}$. Also, for most values of α, $\Re s = 3/2$ is the natural boundary of $F_{2,H,\alpha}(s)$.

Theorem 2. If $b, q > 0$ and $(b, q) = 1$, then

$$F_{2,q,b,H,\alpha}(s) = \sum_{\substack{m_1, m_2 \geq 1 \\ m_1 m_2 \equiv b \pmod q}} \frac{\Lambda(m_1)\Lambda(m_2)}{(m_1 + m_2)^s} H(\alpha \log(m_1 + m_2))$$

is holomorphic in $\Re s > 3/2$.

Outline of the proof of Theorem 1. The first assertion in Theorem 1 follows from an application of the Mellin-Barnes integral formula, which is:

$$(1 + \lambda)^{-s} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s - z)\Gamma(z)}{\Gamma(s)} \lambda^{-z} \, dz,$$

where $s, \lambda \in \mathbb{C}$, $\lambda \neq 0$, $|\arg \lambda| < \pi$, and $0 < c < \Re s$.

In the second assertion, we can show that, for each $n \in \mathbb{Z}$ and $\gamma \in \mathcal{I}$, $F_{2,H,\alpha}(s)$ has a singularity at each point

$$P_{\gamma,n} = \frac{3}{2} + i(\gamma + 2\pi\alpha n).$$

Because the sequence $\{\gamma/(2\pi\alpha)\}$ is uniformly distributed, the set

$$D_\alpha = \{\gamma + 2\pi\alpha n : n \in \mathbb{Z}, \gamma \in \mathcal{I}\}$$

is dense in $\mathbb{R}$. Next, we show the behavior of $F_{2,H,\alpha}(s)$ near the points $P_{\gamma,n}$. Note that we may write

$$F_{2,H,\alpha}(s) = \sum_{m \in \mathbb{Z}} c_m \phi_2(s - 2\pi\alpha i m).$$

Now fix $n \in \mathbb{Z}$, $\gamma \in \mathcal{I}$, and $\kappa = \gamma + 2\pi\alpha n$. Also, fix $0 < \eta < 1$. We write

$$\phi_2(z - 2\pi\alpha i n) = \frac{a_{-1}}{z - (3/2 + i\kappa)} + \sum_{k=0}^{\infty} a_k \left(z - \left(\frac{3}{2} + i\kappa\right)\right)^k$$

for the Laurent series expansion of $\phi_2(z)$ at $z = 3/2 + i\kappa$.

Then

$$\phi_2(\eta + 3/2 + i\gamma) = \frac{a_{-1}}{\eta} + \sum_{k=0}^{\infty} a_k \eta^k.$$

Note that

$$F_{2,H,\alpha}(s) = h_n(s) + c_n\phi_2(s - 2\pi\alpha i n),$$

where

$$h_n(s) = \sum_{\substack{m\in\mathcal{Z}\\ m\neq n}} c_m\phi_2(s - 2\pi\alpha i m).$$

Putting $s = \eta + 3/2 + i\kappa$, we obtain

$$h_n(\eta + 3/2 + i\kappa) = \sum_{\substack{m\in\mathcal{Z}\\ m\neq n}} c_m\phi_2(\eta + 3/2 + i(\kappa - 2\pi\alpha m)).$$

It remains to estimate the sum.

Applying the Mellin-Barnes integral to $\phi_2(\eta+3/2+i(\kappa-2\pi\alpha m))$, we find that it equals

$$\frac{M\left(\eta+\frac{1}{2}+i\left(\kappa-2\pi\alpha m\right)\right)}{\eta+\frac{1}{2}+i(\kappa-2\pi\alpha m)}-\log(2\pi)M\left(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m)\right)$$
$$+\frac{1}{2\pi i}\int_{(-\delta)}\frac{\Gamma\left(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m)-z\right)\Gamma(z)M(z)M\left(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m)-z\right)}{\Gamma\left(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m)\right)}dz$$
$$-\sum_{\rho\in I}\frac{\Gamma\left(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m)-\rho\right)\Gamma(\rho)M\left(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m)-\rho\right)}{\Gamma\left(\eta+\frac{3}{2}+i(\kappa-2\pi\alpha m)\right)}.$$

Then employing Sterling's formula, summation by parts, and

$$M(s)=-\frac{\zeta'(s)}{\zeta(s)}=-\sum_{\substack{\rho\\|\gamma-t|<1}}\frac{1}{s-\rho}+O(\log|t|+2),$$

which holds uniformly for $s=\sigma+it$, where $-1\leq\sigma\leq 2$, to the four quantities above, we find that

$$h_n(\eta+3/2+i\kappa)\leq C(\epsilon)+\frac{\epsilon\beta}{\eta}+D,$$

where $C(\epsilon)$, β, and D are constants such that $C(\epsilon)$ is independent of η, β is independent of ϵ and η, and D is independent of η.

Outline of the proof of Theorem 2. Simply note that

$$F_{2,q,b,H,\alpha}(s) = \sum_{\substack{m,k \geq 1 \\ mk \equiv b \pmod q}} \frac{\Lambda(m)\Lambda(k)}{(m+k)^s} H(\alpha \log(m+k))$$

$$= \sum_{\chi \pmod q} \overline{\chi}(b) F_{2,\chi,H,\alpha}(s),$$

where

$$F_{2,\chi,H,\alpha}(s) = \sum_{n \in \mathbb{Z}} c_n \phi_2(s - 2\pi i \alpha n, \chi).$$

Here

$$\phi_2(s,\chi) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Lambda(k)\Lambda(m)\chi(k)\chi(m)}{(k+m)^s} = \sum_{n=1}^{\infty} \frac{G_2(n,\chi)}{n^s}$$

with

$$G_2(n,\chi) = \sum_{k+m=n} \Lambda(k)\Lambda(m)\chi(k)\chi(m).$$

Theorem 3. Assume the Generalized Riemann Hypothesis. Let b, $q > 0$, and $r \geq 2$ be any integers with $(b, q) = 1$. Also, let $H \in \mathcal{C}^r(\mathbb{R})$ be periodic with period 1, and $\alpha > 0$. Then

$$F_{r,q,b,H,\alpha}(s) = \sum_{\substack{m,\ldots,m_r \geq 1 \\ m_1 \cdots m_r \equiv b \pmod q}} H(\alpha \log(m_1 + \ldots + m_r)) \times \frac{\Lambda(m_1) \cdots \Lambda(m_r)}{(m_1 + \ldots + m_r)^s}.$$

is analytic in the half plane $\Re(s) > r - 1/2$, except for simple poles at $s = r + 2\pi i \alpha n,\ n \in \mathbb{Z}$.

As an immediate corollary, we have the following result.

Corollary 4. Assume the Generalized Riemann Hypothesis. Let $(b_1, q) = 1$, $(b_2, q) = 1$, and $H \in \mathcal{C}^r(\mathbb{R})$ be as in Theorem 3. Then $F_{r,q,b_1,H,\alpha}(s) - F_{r,q,b_2,H,\alpha}(s)$ is analytic on the half plane $\Re(s) > r - 1$.