Finite Euler Products and the Riemann Hypothesis

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Graduate Workshop on Zeta functions, L-functions and their Applications

Outline

1. Approximations of $\zeta(s)$
2. A Function Related to $\zeta(s)$ and its Zeros
3. The Relation Between $\zeta(s)$ and $\zeta_X(s)$

I. Approximations of $\zeta(s)$

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Example

When $X = t$ we have

$$\zeta(s) = \sum_{n \le t} n^{-s} + O(t^{-\sigma}) \qquad (\sigma > 0)\,.$$

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The Lindelöf Hypothesis is true if and only if

$$\zeta(s) = \sum_{n \leq X} \frac{1}{n^s} + O(X^{1/2-\sigma}|t|^{\epsilon})$$

for $\frac{1}{2} \leq \sigma \ll 1$ *and* $1 \leq X \leq t^2$.

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Thus, on LH even short truncations approximate $\zeta(s)$ well in $\sigma > 1/2$.

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These are not equal if X is small relative to T.

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$\Lambda(n) = \log p$ if $n = p^k$, otherwise $\Lambda(n) = 0$. We "smooth" the Λ's and call the result $P_X(s)$.

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Remember

$$P_X(s) \approx \prod_{p \le X^2} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Definition of $Q_X(s)$

We also write

$$Q_X(s) = \exp\left(\sum_{\rho} F_2((s-\rho)\log X)\right) \cdot \exp\left(\sum_{n=1}^{\infty} F_2((s+2n)\log X)\right) \cdot \exp\left(F_2((1-s)\log X)\right)$$

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For z large $F_2(z)$ is small. For z near 0

$$F_2(z) \sim \log(c\,z)\,.$$

A Hybrid Formula for $\zeta(s)$

It follows that in the critical strip away from $s = 1$

$$Q_X(s) \approx \prod_{|\rho - s| \leq 1/\log X} \Big(c\,(s - \rho) \log X \Big)$$

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Theorem

Assume RH. Let $2 \le X \le t^2$ and $\frac{1}{2} + \frac{C\log\log t}{\log X} \le \sigma \le 1$ with $C > 1$. Then

$$\zeta(s) = P_X(s)\Big(1 + O\big(\log^{(1-C)/2} t\big)\Big).$$

Conversely, this implies $\zeta(s)$ has at most a finite number of complex zeros in this region.

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The last estimate also shows that if $\sigma < 1/2$, then infinitely often in t

$$P_X(s) \gg \exp\left(\frac{X^{1-2\sigma}}{\sqrt{\log X}}\right), \quad \text{which is very large.}$$

II. A Function Related to $\zeta(s)$ and its Zeros

Deficiency of the Sum Approximation on $\sigma = 1/2$

On LH (and so on RH) we saw that for $\frac{1}{2} < \sigma \leq 1$ fixed,

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Here $\quad \chi(s) = \pi^{s-1/2}\Gamma(1/2 - s/2)/\Gamma(s/2).$

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$$\zeta(\tfrac{1}{2}+it) = \sum_{n\leq\sqrt{t/2\pi}} \frac{1}{n^{\frac{1}{2}+it}} + \chi(\tfrac{1}{2}+it) \sum_{n\leq\sqrt{t/2\pi}} \frac{1}{n^{\frac{1}{2}-it}} + o(1)\,.$$

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$$\chi(s)P_X(1-s) = \Omega\big(t^{1/2-\sigma}\exp(X^{\sigma-\frac{1}{2}}/\log X)\big)\,,$$

Deficiency of the Euler Product Approximation on $\sigma = 1/2$

How much is the Euler product approximation

$$\zeta(s) = P_X(s)\big(1 + o(1)\big)$$

off by as σ approaches $1/2$?

A tempting guess is that for some range of X

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Lemma

In $0 \le \sigma \le 1, |t| \ge 10, \ \ |\chi(s)| = 1$ *if and only if* $\sigma = 1/2$. *Furthermore,*

$$\chi(s) = \left(\frac{t}{2\pi}\right)^{1/2-\sigma-it} e^{it+i\pi/4}\Big(1 + O(t^{-1})\Big).$$

The Riemann Hypothesis for $\zeta_X(s)$

Theorem

All of the zeros of

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in $0 \leq \sigma \leq 1$ *and* $|t| \geq 10$ *lie on* $\sigma = 1/2$.

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Proof.

$$\zeta_X(s) = P_X(s) \left(1 + \chi(s) \frac{P_X(\bar{s})}{P_X(s)} \right).$$

Also, $P_X(s)$ is never 0. Thus, if s is a zero, $|\chi(\sigma + it)| = 1$. By the lemma, when $|t| \geq 10$ this implies that $\sigma = 1/2$. □

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This vanishes if and only if

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Detecting Zeros of $\zeta_X(s)$

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Lower Bound for the Number of Zeros

So

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How large can the sum be?

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- $\Phi(t) = (1/2+\epsilon)\log t/\log\log t$ is admissible on RH.

The Sum on RH

Conjecture (Farmer, G., Hughes)

$\Phi(t) = \sqrt{(1/2+\epsilon)\log t \log\log t}$ *is admissible, but*
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This is $\ll \Phi(t)$ if $X \ge \exp(c\log t/\Phi(t))$ for some $c > 0$.
(Same bound as for $S(t)$!)

Extra Solutions

If $F_X(t)$ is not monotonically increasing, there could be “extra” solutions of

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On RH the sum is $\ll \Phi(t)\log X$.

Thus, on RH there is a positive constant C, such that $F_X(t)$ is strictly increasing if

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Theorem

If $X \le t^{o(1)}$, then

$$N_X(t) \sim \frac{t}{2\pi}\log\frac{t}{2\pi}.$$

Simple Zeros of $\zeta_X(s)$

$1/2 + i\gamma$ is a simple zero of $\zeta_X(s)$ if $\zeta_X(1/2 + i\gamma) = 0$, but $\zeta_X'(1/2 + i\gamma) \neq 0$.

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This vanishes at $1/2 + i\gamma$ if and only if $F_X'(\gamma) = 0$.

The Number of Simple Zeros When X is Small

Recall that if X is not too large,

$$F_X'(t) = \frac{1}{2\pi} \log \frac{t}{2\pi} - \frac{1}{\pi} \sum_{n \le X^2} \frac{\Lambda_X(n) \cos(t \log n)}{n^{1/2}} + O\left(\frac{1}{t^2}\right) > 0.$$

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Thus we have the

Theorem

Assume RH. There is a constant $C > 0$ such that if $X < \exp\left(C\log t/\Phi(t)\right)$, all the zeros of $\zeta_X(1/2+it)$ with imaginary part ≥ 10 are simple.

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$$F_X'(t) = \frac{1}{2\pi} \log \frac{t}{2\pi} - \frac{1}{\pi} \sum_{n \le X^2} \frac{\Lambda_X(n) \cos(t \log n)}{n^{1/2}} + O\left(\frac{1}{t^2}\right) > 0.$$

Thus we have the

Theorem

Assume RH. There is a constant $C > 0$ such that if $X < \exp\left(C \log t / \Phi(t)\right)$, all the zeros of $\zeta_X(1/2 + it)$ with imaginary part ≥ 10 are simple.

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If $X \leq \exp\left(o(\log^{1-\epsilon} t)\right)$, then $\zeta_X(1/2+it)$ has $\sim T/2\pi\log\left(T/2\pi\right)$ simple zeros up to height T.

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- But even when X is very large, the odds that $F_X'(\gamma) = 0$ are quite small.

III. The Relation Between $\zeta(s)$ and $\zeta_X(s)$

Comparing $\zeta(s)$ and $\zeta_X(s)$

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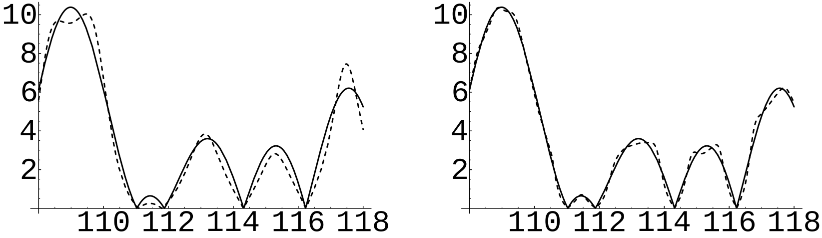

Figure: Graphs of $2|\zeta(1/2 + it)|$ (solid) and $|\zeta_X(1/2 + it)|$ (dotted) near $t = 114$ for $X = 10$ and $X = 300$, respectively.

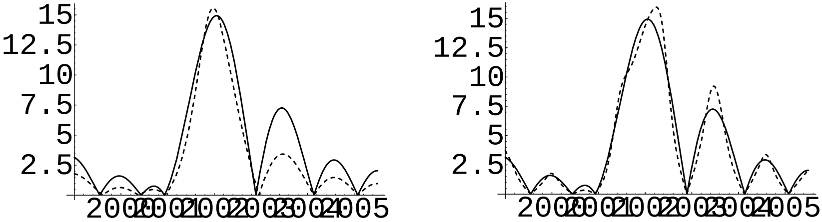

Figure: Graphs of $2|\zeta(\frac{1}{2}+it)|$ (solid) and $|\zeta_X(\frac{1}{2}+it)|$ (dotted) near $t = 2000$ for $X = 10$ and $X = 300$, respectively.

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Why?

The Heuristic Reason Why $|\zeta_X(1/2+it)| \approx 2\,|\zeta(1/2+it)|$

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In fact, this suggests that $\zeta_X(1/2+it) \approx 2\,\zeta(1/2+it)$.

Why Zeros of $\zeta_X(s)$ and $\zeta(s)$ are Close

$$F_X(t) = -\frac{1}{2\pi}\arg\chi(1/2+it) + S(t) - \frac{1}{\pi}\mathrm{Im}\sum_{\gamma} F_2(i(t-\gamma)\log X) + E$$

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$$\ll_{\mathcal{I}} \frac{1}{\log^2 X}\sum_{\gamma}\frac{1}{(t-\gamma)^2} \to 0 \qquad \text{as } X \to \infty.$$

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- $\zeta_X(1/2+it)$ *has no zeros in* $\mathcal{I}$ *for* X *sufficiently large.*

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Later Jon Keating and Eugene Bogomolny used $\zeta_{t/2\pi}(1/2 + it)$ as a heuristic tool for calculating the pair correlation function of the zeros of $\zeta(s)$.

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- The general problem is to see what further insights we can gain into the behavior of $\zeta(s)$ and other L-functions from these models.
- Study the number of zeros of $\zeta_X(s)$ and the number of simple zeros when X is large, say $X = t^\alpha$.
- $\zeta_X(s)$ approximates $\mathcal{F} = \zeta(s) + \chi(s)\zeta(\bar{s})$ well in $\sigma > 1/2 + \log\log t/\log X$ and on $\sigma = 1/2$ when X is large. What about in between?

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Together with Jon Keating, we are beginning to determine the outlines of a theory of such moments, even when X is much larger than T.