2009 Graduate Workshop on Zeta functions, L-functions and their Applications

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June 9, 2009

Contents

Abstract

These notes were TeXed in real-time by Steven J. Miller; all errors should be attributed to the typist.

1 Hundley: Integral Representations: What we've found, what we're looking for and why we're looking there

http://www.williams.edu/go/math/sjmiller/public_html/ntandrmt/talks/UtahWorkshopNotes.pdf

NOTE: these notes are much rougher than the others, due to lateness of the day. All omissions should be attributed to the typist.

I want to mention a copy of major sources for everything I'm going to say: surveys of Bump in the Selberg volume and the Rallis volume. Also a lot of this was picked up from Ginzburg.

1.1 Langlands (Partial) L-functions

Data:

- $\pi \cong \otimes'_v \pi_v$ irreducible cuspidal (generic) representation, G a quasi-split connected reductive algebraic group, $\mathbf{A} = \prod'_v F_v$ adele ring. Such π's can be obtained from classical Dirichlet, Hecke characters, modular forms, Maass forms, Siegel (not gen.) modular forms.
- S: finite set of places including $v|\infty$ and all v where π_v is not unramifiied principal series.
- r: finite dimensional representation of ${}^L G$.

$$L^S(s, \pi, r) \;=\; \prod_{v \in S} \det(I - q_V^S r(t_{\pi_v}))^{-1}, \tag{1}$$

with $s \in \mathbb{C}$ and $q_v = \#(\mathcal{O}_v/p_v)$ and ${}^L G \ni t_{\pi_v} \leftrightarrow \pi_v$.

1.2 Integral Representations

- Different usage of representation.
- Express given L-function as integral using automorphic form from π.
- Failing that, find an integral which when evaluated gives some L-functions.

Classical examples:

- Hecke: $\int_0^\infty f(iy) y^s dy/y$ with $F = \mathbb{Q}$, $G = \mathrm{GL}(2)$, π generated by f and ${}^L G = \mathrm{GL}(2, \mathbb{C})$, $r =$ std representation.

- Rankin-Selberg: $\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}} f_1(z)f_2(z)E(z,s)dxdy/y^2$ with $E(z,s) = \sum_{\mathbb{Z}^2-(0,0)} y^s/|mz+n|^{2s}$. Here $F = \mathbb{Q}$, $G = \mathrm{GL}(2)^2$, π generated by $f_1 \cdot f_2$, ${}^LG = \mathrm{GL}(2,\mathbb{C})^2$, $r(g,h) \cdot X = gX + h$.
- $\int_0^1 E(z,s)dx = \cdots$, involves $\zeta(2s)$ and $\zeta(2s-1)$.

1.3 Recent Examples

- Dutta-She: uses four-fold cover of $\mathrm{GL}(3)$. Doesn't have a unique Whittaker model.
- Ishii-Moriyama: Archimedean (introduced by Novodvorsky).
- Pitale-Schmidt: $\mathrm{GSp}_4 \times \mathrm{GL}_2$: non-generic (after Furusawa).
- Qin: $G = U(n,n)$.
- Soudry: book (in progress for some time): has some general constructions for GL cross a classical group.
- Wambach: published details of above.
- Ginzburg: Doubling method for G_2
- Ginzurg-Hundley: local computations get nasty. Here $G = \mathrm{GL}_5$, ${}^LG = \mathrm{GL}_5(\mathbb{C})$, $r = \mathrm{Ad}$, $G = \mathrm{Spin}_{11}$, ${}^LG = \mathrm{Sp}_{10}(\mathbb{C})$, $r = \Lambda_0^2$.
- Hundley: $G = \mathrm{SU}_{2,1}$....

There are constructions where you get the adjoint for GL_4. These use Eisenstein series. Also constructions for GL_3, use the same Eisenstein series on GL_2. A third construction studied by Jiang-Rallis, get a ratio of Dedekind zeta functions.

1.4 Where are we looking and why?

Goal: find Eulerian integrals.

Paradigm: They come from uniqueness principles. The classical Rankin-Selberg can be considered as coming from a trilinear form. Can prove that trilinear form is unique. Bernstein-Reznikov were trying to compute

$$\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}} f_1(z)f_2(z)f_3(z)dxdy/y^2 \tag{2}$$

from uniqueness of trilinear forms without unfolding.

Sakellaridis put up a very interesting preprint on the arxiv about a week ago. Extremely relevant to this. He seems to have a way to generate a lot of multiplicity one statements through the theory of spherical varieties, recovering many of the best examples. He is not claiming to have all of them for everything called Rankin-Selberg. Gets three new integrals out of this, promise of some conceptual picture of where many of these come from.

1.5 Ginzburg's dimension heuristics

$P = MU$ parabolic, M acts on characters of U via rational representation defined over F. Some cases U has generalized Heisenberg structure. Characters correspond to nilpotent elements of Lie algebra. Fourier coefficients correspond to nilpotent orbits.

Heuristic: the sum of the GK dimensions of the representations involved must equal the dimension of the domain of integration. Ask: do there exist a parabolic $Q = LV$ and a representation τ of L with GK dimension such that GK dimension plus GK dimension τ plus $\dim V$ equals $\dim U$ plus $\dim H_{\psi_u^0}$.