The Pair Correlation of Zeros of $\xi^{(\kappa)}(s)$

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Outline

Riemann's xi-Function and Zeros of Riemann's Zeta-Function

To study the zeros of $\zeta(s)$, we introduce the Riemann's xi-funtion, which is defined as

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

It is an entire function with the functional equation

$$\xi(1-s) = \xi(s),$$

and $\xi(s)$ has the same complex zeros as $\zeta(s)$.

Number of Zeros Up To Height T

Define $N(T)$ to be the number of zeros of $\zeta(s)$ up to height T, that is,

$$N(T) = \#\{\rho = \beta + i\gamma : 0 \le \beta \le 1, 0 \le \gamma \le T\}.$$

Then we have

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

From this we find that the average space between consecutive zeros is

$$\frac{T}{N(T)} \sim \frac{2\pi}{\log T}.$$

The Horizontal Distribution of the Zeros: the Riemann Hypothesis

From the functional equation $\xi(1-s) = \xi(s)$, we observe that the zeros are symmetrically distributed about the real axis and the critical line $\sigma = 1/2$. With this picture in mind we state the following famous conjectures.

conjecture (The Riemann Hypothesis)

For each zero $\rho = \beta + i\gamma$ of $\zeta(s)$, we have $\beta = 1/2$.

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conjecture (The Riemann Hypothesis)

For each zero $\rho=\beta+i\gamma$ of $\zeta(s)$, we have $\beta=1/2$.

conjecture (The Simple Zero Conjecture)

All zeros of $\zeta(s)$ are simple.

The Horizontal Distribution of the Zeros of $\xi^{(\kappa)}(s)$

If the Riemann Hypothesis is true, all the zeros of $\xi(s)$ have real part one-half. Since the zeros of the derivative of an entire function are contained in the convex hull of the zeros of the function, it then also follows that all the zeros of $\xi^{(\kappa)}(s)$ have real part one-half.

The Vertical Distribution of the Zeros: Montgomery's Function

To study the vertical distribution of zeros of $\zeta(s)$ that lie on the critical line, Montgomery introduced the following function

$$F(\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma\leq T; 0<\gamma'\leq T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma'),$$

where α and $T \geq 2$ are real. Here $w(u)$ is a suitable weight function, $w(u) = 4/(4+u^2)$.

Montgomery's Main Result

His main result is

$$F(\alpha) = (1 + o(1))T^{-2|\alpha|}\log T + \alpha + o(1)$$

uniformly for $0 \leq |\alpha| \leq 1 - \epsilon$ as T tends to infinity.

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For $|\alpha| \geq 1$, he conjectured that

$$F(\alpha) = 1 + o(1)$$

holds uniformly in bounded intervals.

Applications of Montgomery's Function

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Applications of Montgomery's Function

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$$\liminf(\gamma_{n+1} - \gamma_n)(\log \gamma_n/2\pi) \leq 0.68.$$

He also deduced an estimate for the percentage of simple zeros

$$\sum_{0<\gamma\leq T;\rho \text{ simple}} 1 \geq (\frac{2}{3} + o(1))\frac{T}{2\pi} \log T,$$

where the sum is over simple zeros up to height T.

The Vertical Distribution of the Zeros: Montgomery's Conjecture

From his results on the function $F(\alpha)$, Montgomery was led to the following conjecture.

conjecture (Pair Correlation Conjecture (PCC))

For any fixed $\alpha > 0$,

$$N(T,\alpha) := \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{0<\gamma,\gamma'\leq T \\ 0<\gamma-\gamma'\leq\frac{2\pi\alpha}{\log T}}} 1 \sim \int_0^\alpha 1 - \left(\frac{\sin\pi u}{\pi u}\right)^2 du.$$

Connection with Random Matrix Theory

Montgomery's work led to the remarkable conclusion that the zeros of $\zeta(s)$ might be distributed like the eigenvalues of large random matrices. The density function $1-\left(\frac{\sin \pi u}{\pi u}\right)^2$ in PCC happens to coincide with the pair correlation function for the Gaussian Unitary Ensemble (GUE).

Farmer and Gonek's Results on $\xi'(s)$

Recently, Farmer and Gonek obtained analogous results for $\xi'(s)$. It is observed that the distribution is no longer GUE, and the zeros are more evened out.

For instance, they obtained that more than 85.84% of the zeros of $\xi'(s)$ are simple, and a positive proportion of gaps between zeros of $\xi'(s)$ are less than 0.91 times the average spacing.

The Generalized Montgomery Function $F_\kappa(\alpha, T)$

To study the pair correlation of zeros of $\xi^{(\kappa)}(s)$, we define the function $F_\kappa(\alpha, T)$ as

$$F_\kappa(\alpha, T) = \frac{2\pi}{T\log T} \sum_{0<\gamma_\kappa,\gamma'_\kappa\le T} T^{i\alpha(\gamma_\kappa-\gamma'_\kappa)} w(\gamma_\kappa-\gamma'_\kappa),$$

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where α and $T \ge 2$ are real. Here $w(u)$ is the same weight function as in Montgomery's function. Notice that if we take κ to be 0, it reduces to Montgomery's case.

Main Result

Theorem

Assume the Riemann Hypothesis, we have

$$F_\kappa(\alpha, T) = \frac{2\pi}{T\log T}\sum_{0<\gamma_\kappa,\gamma'_\kappa\le T} T^{i\alpha(\gamma_\kappa-\gamma'_\kappa)} w(\gamma_\kappa-\gamma'_\kappa)$$
$$= (1+o(1))T^{-2|\alpha|}\log T + \sum_{i=1}^{2\kappa B+1} C_{\kappa i}|\alpha|^i + o_{\kappa,B}(1).$$

uniformly for $0<|\alpha|\le 1-\epsilon$.

Where the coefficients $C_{\kappa i}$ are computable constants.

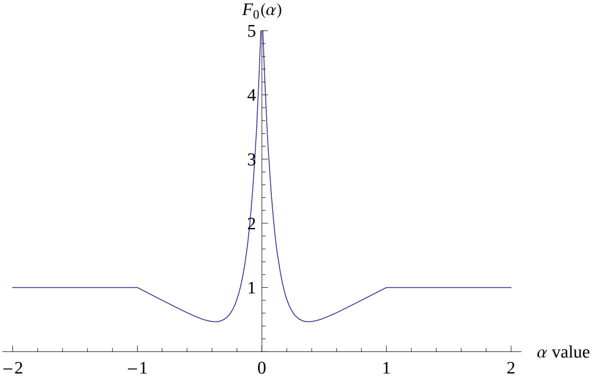
Tables of $C_{\kappa i}$

Numerical Approximation

	i=1	2	3	4	5	6	7	8	9	10	11
κ=1	1.00	−4.00	4.00	0	1.33	0	0.356	0	0.0762	0	0.0135
2	1.00	−4.00	4.00	−16.0	28.0	16.0	12.1	−11.4	−1.65	−0.440	4.25
3	1.00	−4.00	4.00	−16.0	66.4	−149.	258.	240.	6.03	−109.	130.
4	1.00	−4.00	4.00	−16.0	66.4	−224.	602.	-1.21×10^3	1.50×10^3	2.00×10^3	−237.
5	1.00	−4.00	4.00	−16.0	66.4	−224.	766.	-2.00×10^3	4.93×10^3	-9.20×10^3	1.33×10^4
6	1.00	−4.00	4.00	−16.0	66.4	−224.	766.	-2.32×10^3	6.51×10^3	-1.63×10^4	3.78×10^4
7	1.00	−4.00	4.00	−16.0	66.4	−224.	766.	-2.32×10^3	7.19×10^3	-1.98×10^4	5.36×10^4
8	1.00	−4.00	4.00	−16.0	66.4	−224.	766.	-2.32×10^3	7.19×10^3	-2.11×10^4	6.04×10^4
9	1.00	−4.00	4.00	−16.0	66.4	−224.	766.	-2.32×10^3	7.19×10^3	-2.11×10^4	6.32×10^4
10	1.00	−4.00	4.00	−16.0	66.4	−224.	766.	-2.32×10^3	7.19×10^3	-2.11×10^4	6.32×10^4
11	1.00	−4.00	4.00	−16.0	66.4	−224.	766.	-2.32×10^3	7.19×10^3	-2.11×10^4	6.32×10^4

Graphs of $F_\kappa(\alpha, T)$

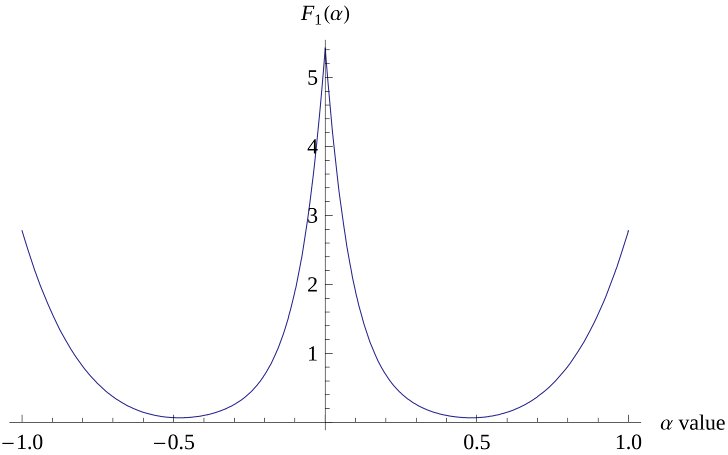
Using this table, we obtained several graphs of $F_\kappa(\alpha, T)$ for different values of κ:



$\kappa = 0$

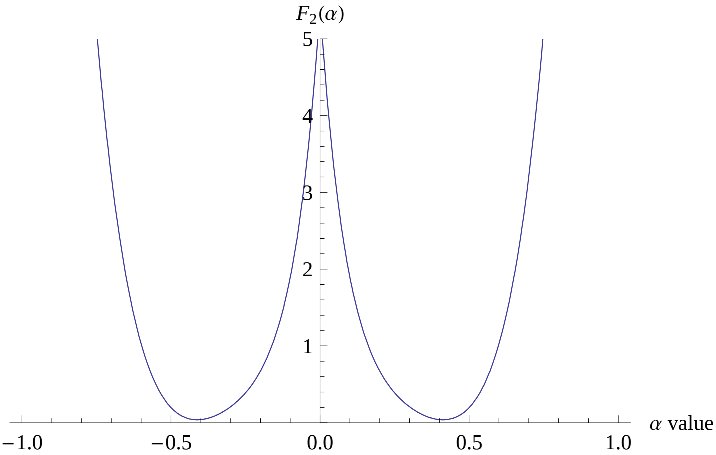
Graphs of $F_\kappa(\alpha, T)$

$\kappa = 1$



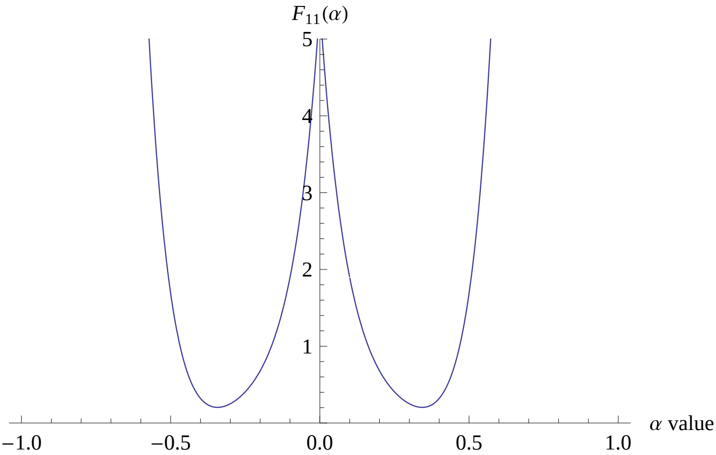
Graphs of $F_\kappa(\alpha, T)$

$\kappa = 2$



Graphs of $F_\kappa(\alpha, T)$

$\kappa = 11$



The Picket Fence Model

It is predicted by Farmer and Rhoades that the zeros of $\xi^{(\kappa)}(s)$ tend to even out when we take high derivatives. In order to see this, we compare our results with the Picket Fence Model.

It is known that the number of zeros of $\xi^{(\kappa)}(s)$ up to height T is

$$N^{(\kappa)}(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O_\kappa(\log T),$$

and the average space between consecutive zeros is

$$\frac{T}{N^{(\kappa)}(T)} \sim \frac{2\pi}{\log T}.$$

The Picket Fence Model

Recall that the generalized Montgomery function $F_\kappa(\alpha, T)$ is defined as

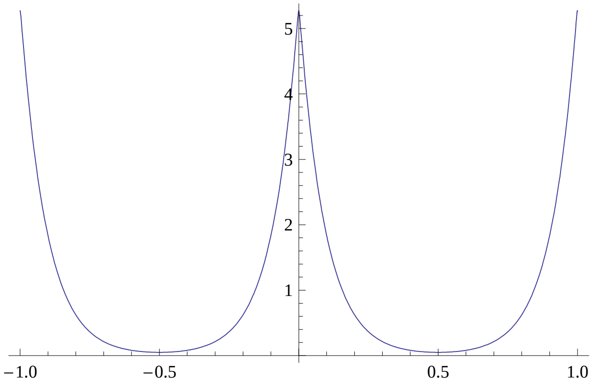
$$F_\kappa(\alpha, T) = \frac{2\pi}{T \log T} \sum_{0<\gamma_\kappa,\gamma'_\kappa \le T} T^{i\alpha(\gamma_\kappa-\gamma'_\kappa)} w(\gamma_\kappa - \gamma'_\kappa).$$

If the zeros were equally spaced, then we would have

$$F(\alpha) = \frac{1}{N} \sum_{0<m,n\le N} e^{2\pi i\alpha(m-n)} \frac{4}{4+(m-n)^2 \frac{4\pi^2}{\log^2 N}}.$$

Graph of the Generalized Montgomery Function of the Picket Fence Case

Generalized Montgomery Function of the Picket Fence Case



Applications of Our Result

Two direct applications of our results on $F_\kappa(\alpha, T)$ will be to obtain results on small gaps between the zeros of $\xi^{(\kappa)}(s)$ and on the percentage of simple zeros, as in Montgomery's paper.

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For the simple zeros, let λ_κ denote the proportion of zeros of $\xi^{(\kappa)}(s)$ that are on $\sigma = 1/2$ and are simple. Our result suggests that $\lambda_2 > 0.9544$, and $\lambda_3 > 0.9774$ (on RH).

The Application to Small Gaps

Applying our results to small gaps between zeros, our results suggest that a positive proportion of gaps between zeros of $\xi'(s)$ are less than 0.8967 times the average spacing. For $\xi''(s)$, the size is 0.9283, and for $\xi'''(s)$, the size is 0.9415.

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Therefore, our results indicate that the zeros tend to even out when we take high derivatives, which is consistent with the work of Farmer and Rhoades.

Lower Bounds For the Percentage of Simple Zeros

For the case $\kappa = 0$, our result agrees with Montgomery's result and for the case $\kappa = 1$, our result shows that the percentage is more than 0.8584 (on RH), which confirms the result of Farmer and Gonek .

For higher cases of κ, we can not prove that the tail of the function $F_\kappa(\alpha)$ is small. However for $\kappa = 2, 3$, empirically we observe that the $C_{\kappa i}$ seem to stabilize and are small after certain point.

The Application to Simple Zeros For $\kappa = 2, 3$

proposition

Assuming the coefficients after the first 11 terms are negligible, we have:
For $\kappa = 2$, more than 95.44% of the zeros of $\xi''(s)$ are simple.
For $\kappa = 3$, more than 97.74% of the zeros of $\xi'''(s)$ are simple.

Both results are on RH.

The Application to Simple Zeros For $\kappa = 2, 3$

For $\kappa = 2$, the coefficients are relatively small so our result is likely to be close to the exact value.

On the other hand, for $\kappa = 3$, the coefficients are not settling down yet. Therefore, we do not know how close our result is to the exact value.

Brian's Result

Let λ_κ denote the proportion of zeros of $\xi^{(\kappa)}(s)$ that are on $\sigma = 1/2$ and are simple. The best currently available bounds are $\lambda_0 > 0.4$, $\lambda_1 > 0.7869$, $\lambda_2 > 0.9314$, $\lambda_3 > 0.9666$, $\lambda_4 > 0.9799$, and $\lambda_5 > 0.9863$ (Recall our results $\lambda_2 > 0.9544$, and $\lambda_3 > 0.9774$). This was obtained by Brian Conrey in 1983.

Cases When $\kappa > 3$

For $\kappa > 3$, we can not obtain a reasonable result for the lack of data. The amount of calculations needed exceeds the capability of a personal computer and the coefficients we obtained are large, which suggests that the coefficients we ignored make a contribution.

Pair Correlation of Zeros of Distinct Derivatives

By similar arguments, we can apply our method on pair correlation of zeros of distinct derivatives $\xi^{(\iota)}(s)$ and $\xi^{(\kappa)}(s)$. Define the generalized Montgomery function

$$F_{\iota,\kappa}(\alpha, T) = \left(\frac{T}{2\pi}\log\frac{T}{2\pi}\right)^{-1} \sum_{0<\gamma_\iota,\gamma'_\kappa\le T} T^{i\alpha(\gamma_\iota-\gamma'_\kappa)} w(\gamma_\iota-\gamma'_\kappa)\,, \quad (1)$$

where the sum is over pairs of ordinates of the zeros of $\xi^{(\iota)}(s)$ and $\xi^{(\kappa)}(s)$. Notice that in the case $\iota = 0, \kappa = 1$, the "multiple zeros" $\gamma_0 = \gamma'_1$ correspond to multiple zeros of $\zeta(s)$.

Main Result for $F_{\iota,\kappa}(\alpha, T)$

For the function $F_{\iota,\kappa}(\alpha, T)$, we have the following result:

Theorem

Assume the Riemann Hypothesis. Let B be an arbitrary large positive integer, and orders of derivatives ι and κ be nonnegative integers. Then there are constants $C_{\iota\kappa m}$ such that

$$\begin{aligned} F_{\iota,\kappa}(\alpha, T) &= \frac{2\pi}{T\log T}\sum_{0<\gamma_\iota,\gamma'_\kappa\le T} T^{i\alpha(\gamma_\iota-\gamma'_\kappa)}w(\gamma_\iota-\gamma'_\kappa) \\ &= (1+o(1))T^{-2|\alpha|}\log T + \sum_{m=1}^{(\iota+\kappa)B+1} C_{\iota\kappa m}|\alpha|^m + o(1) \end{aligned}$$

uniformly for $0<|\alpha|<1$.

Corollary

In particular, when $\iota = 0$, we have the following corollary,

Corollary

Let B be an arbitrary large positive integer and κ be a positive integer, then we have

$$F_{0,1}(\alpha, T) = (1 + o(1))T^{-2|\alpha|}\log T + |\alpha| - 4|\alpha|^2 + o_B(1),$$

and more generally

$$F_{0,\kappa}(\alpha, T) = (1 + o(1))T^{-2|\alpha|}\log T + |\alpha| - 4|\alpha|^{\kappa+1} + o_{\kappa,B}(1)$$

uniformly for $0 < |\alpha| < 1$.

Upper Bounds For the Percentage of Multiple Zeros

Applying our result on $F_{0,1}(\alpha, T)$ to study the percentage of multiple zeros of $\zeta(s)$, we have the following corollary,

Corollary

The percentage of multiple zeros of $\zeta(s)$ is less than $1/3$.

Notice that this is identical with Montgomery's result.