MEAN VALUE THEOREMS AND THE ZEROS OF THE ZETA FUNCTION

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OUTLINE

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I. What is a mean value theorem?

In general it is an estimate for an average of a function. For example

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + Re^{i\theta})|^2 \, d\theta \, .$$

For a function with a Dirichlet series, $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, the average is typically along a vertical segment:

$$\int_0^T |F(\sigma + it)|^2 \, dt \, ,$$

or

$$\int_0^T F(\sigma + it) \, dt \, .$$

Note:

1) The path of integration might not be in the half–plane of convergence.

2) It is customary not to divide by T.

There are many variants, for example, an average over a discrete set of points:

$$\sum_{r=1}^{R} |F(\sigma_r + it_r)|^2 \qquad\qquad (\sigma_r + it_r \in \mathbb{C}) .$$

Example 1. Take $F(s) = \zeta(s)^k$, $\sigma \geq 1/2$, and k a positive integer. We are interested in the means

$$\begin{aligned} I_k(\sigma, T) = & \int_0^T \left|\zeta(\sigma + it)^k\right|^2 dt \\ = & \int_0^T |\zeta(\sigma + it)|^{2k} \, dt \, . \end{aligned}$$

Example 2. Take $F(s) = (\zeta'(s))^k$, $\mathcal{S}$ a set of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. One can consider the means

$$\sum_{\rho \in \mathcal{S}} |\zeta'(\rho)|^{2k} \, .$$

Example 3. Let

$$F(s) = F_N(s) = \sum_{n=1}^{N} a_n n^{-s}$$

be a Dirichlet "polynomial" of "length" N. One can show that

$$\int_0^T |\sum_{n=1}^{N} a_n n^{-\sigma-it}|^2 dt = (T + O(N \log N)) \sum_{n=1}^{N} \frac{|a_n|^2}{n^{2\sigma}} .$$

This is the "classical mean value theorem for Dirichlet polynomials".

A stronger version, due to H. L. Montgomery and R. C. Vaughan, is

$$\int_0^T |\sum_{n=1}^{N} a_n n^{-\sigma-it}|^2 dt = \sum_{n=1}^{N} \frac{|a_n|^2}{n^{2\sigma}} (T + O(n)) .$$

II. Mean values and zeros.

Mean value estimates are used to study the zeros in a variety of ways. One direct link between them and the zeros of an analytic function is given by

Theorem. (Jensen's Formula) *Let $f(z)$ be analytic for $|z| \leq R$ and suppose that $f(0) \neq 0$. Let $r_1, r_2, \ldots, r_n$ be the moduli of all the zeros of $f(z)$ inside $|z| \leq R$. Then*

$$\log(\frac{|f(0)|R^n}{r_1 r_2 \cdots r_n}) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta\,.$$

That is, the distribution of zeros of $f(z)$ inside the circle is related to the mean of $\log|f(z)|$ on the circle.

There is an analogous result for rectangles that is more useful when working with Dirichlet series.

Theorem. (Littlewood's Lemma) *Let $f(s)$ be analytic and nonzero on the rectangle $\mathcal{C}$ with vertices σ_0, σ_1, $\sigma_1 + iT$, $\sigma_0 + iT$. Then*

$$2\pi \sum_{\rho \in \mathcal{C}} dist(\rho) = \int_0^T \log |f(\sigma_0 + it)| \, dt - \int_0^T \log |f(\sigma_1 + it)| \, dt$$

$$+ \int_{\sigma_0}^{\sigma_1} \arg f(\sigma + iT) \, d\sigma - \int_{\sigma_0}^{\sigma_1} \arg f(\sigma) \, d\sigma \, ,$$

where the sum runs over the zeros ρ of $f(s)$ in $\mathcal{C}$ and $dist(\rho)$ is the distance from ρ to the left edge of the rectangle.

Proof of Littlewood's Lemma.

Let $\mathcal{C}'$ denote the rectangle $\mathcal{C}$ together with the "loops" $\mathcal{L}_\rho$ around each zero ρ (see the figure). Then

$$\int_{\mathcal{C}} \log f(s)\, ds = \int_{\mathcal{C}'} \log f(s)\, ds + \sum_{\rho \in \mathcal{C}} \int_{\mathcal{L}_\rho} \log f(s)\, ds\,.$$

Now $\log f(s)$ is analytic and single–valued in $\mathcal{C}'$, so

$$\int_{\mathcal{C}'} \log f(s)\, ds = 0\,.$$

Therefore,

$$\int_{\mathcal{C}} \log f(s)\, ds = \sum_{\rho \in \mathcal{C}} \int_{\mathcal{L}_\rho} \log f(s)\, ds\,.$$

If $\rho = \beta + i\gamma$, and the circle in $\mathcal{L}_\rho$ has radius r, then

$$\int_{\mathcal{L}_\rho} \log f(s)\, ds = \int_{\sigma_0}^{\beta - r} \log f(\sigma + i\gamma^-)\, d\sigma + \int_0^{2\pi} \log f(re^{i\theta})\, ire^{i\theta} d\theta$$
$$- \int_{\sigma_0}^{\beta - r} \log f(\sigma + i\gamma^+)\, d\sigma\,.$$

The second integral $\to 0$ as $r \to 0^+$. The third integral is

$$\int_{\sigma_0}^{\beta - r} \big(\log f(\sigma + i\gamma^-) + 2\pi i \big)\, d\sigma\, .$$

Hence, as $r \to 0^+$,

$$\int_{\mathcal{L}_\rho} \log f(s)\, ds \to -2\pi i \int_{\sigma_0}^{\beta} d\sigma = -2\pi i(\beta - \sigma_0)\, .$$

Therefore

$$\int_{\mathcal{C}} \log f(s)\, ds = -2\pi i \sum_{\rho \in \mathcal{C}} (\beta - \sigma_0)\, .$$

We may write this as

$$-2\pi i \sum_{\rho \in \mathcal{C}} (\beta - \sigma_0) = \int_0^T \log f(\sigma_1 + it)\, idt - \int_0^T \log f(\sigma_0 + it)\, idt$$

$$+ \int_{\sigma_0}^{\sigma_1} \log f(\sigma)\, d\sigma - \int_{\sigma_0}^{\sigma_1} \log f(\sigma + iT)\, d\sigma\, .$$

The result follows on equating imaginary parts.

Only the first term on the right in Littlewood's Lemma will be significant for us, so we will write

$$2\pi \sum_{\rho\in\mathcal{C}} \mathsf{dist}(\rho) = \int_0^T \log|f(\sigma_0+it)|\,dt - \int_0^T \log|f(\sigma_1+it)|\,dt$$
$$+ \int_{\sigma_0}^{\sigma_1} \arg f(\sigma+iT)\,d\sigma - \int_{\sigma_0}^{\sigma_1} \arg f(\sigma)\,d\sigma\,,$$

as

$$2\pi \sum_{\rho\in\mathcal{C}} \mathsf{dist}(\rho) = \int_0^T \log|f(\sigma_0+it)|\,dt + \mathcal{E}$$

and ignore $\mathcal{E}$.

Usually we cannot estimate the integral directly, so we use the following trick:

$$\begin{aligned}\frac{1}{T}\int_0^T \log|f(\sigma_0+it)|\,dt &= \tfrac{1}{2T}\int_0^T \log(|f(\sigma_0+it)|^2)\,dt \\ &\le \tfrac{1}{2}\log(\tfrac{1}{T}\int_0^T |f(\sigma_0+it)|^2\,dt)\,,\end{aligned}$$

("The average of the $\log$ is $\le$ the $\log$ of the average.")

Note: Our mean values appear!

III. A sample of important estimates.

Recall that

$$I_k(\sigma, T) = \int_0^T |\zeta(\sigma + it)|^{2k} \, dt \, .$$

The case $k = 1$.

If $\sigma > 1/2$,

$$I_1(\sigma, T) = \int_0^T |\zeta(\sigma + it)|^2 \, dt \sim c_1(\sigma) \, T \, ,$$

as $T \to \infty$.

Hardy and Littlewood (1918) proved that if $\sigma = 1/2$, then

$$I_1(1/2, T) \sim T \log T \, .$$

In particular, $|\zeta(\sigma + it)|$ is smaller on average for $\sigma > 1/2$ then for $\sigma = 1/2$.

Since $\zeta(\frac{1}{2} + it)$ has many zeros, we should expect $\zeta(s)$ to be very erratic on $\sigma = \frac{1}{2}$.

The case $k = 2$.

If $\sigma > 1/2$,

$$I_2(\sigma, T) = \int_0^T |\zeta(\sigma + it)|^4 \, dt \sim c_2(\sigma) \, T \, ,$$

as $T \to \infty$.

Ingham (1926) proved that

$$I_2(1/2, T) \sim \frac{T}{2\pi^2} \log^4 T \, .$$

The case $k > 2$.

No asymptotic has yet been proven.

Balasubramanian and Ramachandra have shown that

$$I_k(1/2, T) \gg T \log^{k^2} T \, ,$$

and we expect that

$$I_k(1/2, T) \sim c_k T \log^{k^2} T \, .$$

Conrey and Ghosh conjectured that

$$c_k = \frac{a_k g_k}{\Gamma(k^2+1)} .$$

That is, that

$$I_k(1/2, T) \sim \frac{a_k g_k}{\Gamma(k^2+1)} T \log^{k^2} T .$$

Here

$$a_k = \prod_p \left(\left(1 - \frac{1}{p}\right)^{(k-1)^2} \sum_{r=0}^{k-1} \binom{k-1}{r}^2 p^{-r} \right)$$

and g_k is some (then unknown) constant.

J. Keating and N. Snaith used random matrix theory to conjecture the value of g_k for all k. When k is an integer their conjecture is

Conjeture. (Keating–Snaith)

$$g_k = (k^2!) \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} .$$

Another important mean is

$$(1) \qquad \int_0^T \left|\zeta^{(j)}(\sigma+it)M_N(\sigma+it)\right|^2 dt\,,$$

where

$$M_N(s) = \sum_{1\le n\le N} \frac{\mu(n)}{n^s}(1-\frac{\log n}{\log N})\,.$$

$M_N(s)$ approximates $1/\zeta(s)$ when $\sigma > 1$. This continues for $\sigma \le 1$ in some sense. Hence,

$$\zeta(s)M_N(s)$$

should be tamer than $\zeta(s)$ on $\sigma = \frac{1}{2}$.

General estimates for means like (1) were proved by Conrey, Ghosh, and Gonek with

$$N = T^\theta \qquad \text{and} \qquad \theta < 1/2\,.$$

Later, Conrey used Kloosterman sum techniques to show these formulas also hold for $\theta < 4/7$.

Assuming the Riemann Hypothesis and the Generalized Lindelöf Hypothesis are true, Conrey, Ghosh, and Gonek also proved discrete versions of this, including estimates for sums like

$$\sum_{0<\gamma<T} \left|\zeta'(\rho) M_N(\rho)\right|^2 .$$

Here γ runs over the ordinates of the zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. Again

$$N = T^{\theta} \qquad \text{and} \qquad \theta < 1/2 \, .$$

IV. Application: A simple zero–density estimate.

Let

$$N(\sigma,T)=\sum_{\substack{\rho=\beta+\imath\gamma\\ 0<\gamma\le T\\ \sigma<\beta\le 1}} 1\,.$$

We want an upper bound for $N(\sigma,T)$ when $\frac{1}{2}<\sigma\le 1$ is fixed.

We apply Littlewood's Lemma on the rectangle $\mathcal{C}$ with vertices σ_0, 2, $2+iT$, σ_0+iT, where $\frac{1}{2}<\sigma_0\le 1$ is fixed.

$$\sum_{\rho\in\mathcal{C}} \mathsf{dist}(\rho) = \frac{1}{2\pi}\int_0^T \log(|\zeta(\sigma_0+it)|)\,dt + \mathcal{E}\,.$$

Let σ be a real number with $\sigma_0<\sigma<1$.

On the one hand,

$$\sum_{\rho\in\mathcal{C}} \mathsf{dist}(\rho) \ge \sum_{\substack{\rho\in\mathcal{C}\\ \sigma\le\beta}} \mathsf{dist}(\rho) \ge (\sigma-\sigma_0)N(\sigma,T).$$

On the other hand,

$$\begin{aligned}\frac{1}{2\pi}\int_0^T \log(|\zeta(\sigma_0+it)|)\,dt &= \frac{1}{4\pi}\int_0^T \log(|\zeta(\sigma_0+it)|^2)\,dt\\ &\le \frac{T}{4\pi}\log(\frac{1}{T}\int_0^T |\zeta(\sigma_0+it)|^2)\,dt\end{aligned}$$

by our "trick".

Hence

$$(\sigma - \sigma_0)N(\sigma, T) \leq \frac{T}{4\pi} \log(\frac{1}{T} \int_0^T |\zeta(\sigma_0 + it)|^2) \, dt + \mathcal{E} \, .$$

The integral is

$$I_1(\sigma_0, T) \sim c_1(\sigma_0) \, T \, .$$

Therefore,

$$(\sigma - \sigma_0)N(\sigma, T) \leq \frac{T}{4\pi} \log c_1(\sigma_0) \, ,$$

and

$$N(\sigma, T) \ll T \, .$$

Since $N(T) \sim \frac{T}{2\pi} \log T$, we see that

$$N(\sigma, T)/N(T) = O(\frac{1}{\log T})$$

for any fixed $\sigma > 1/2$.

We may interpret this as saying that only an infinitesimal proportion of the zeros are to the right of any line $Res = \sigma > 1/2$.

This was the first zero–density estimate. It was proved by H. Bohr and E. Landau (1914).

Since then, much stronger results have been proven, typically of the form

$$N(\sigma, T) << T^{\lambda(\sigma)} ,$$

where $\lambda(\sigma) < 1$ and $\lambda(\sigma)$ is decreasing for $\sigma > 1/2$.

V. Application: Levinson's method.

Recall that

$$N(T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \quad 0 < \gamma < T\}$$

$$\sim \frac{T}{2\pi} \log T$$

and let

$$N_0(T) = \#\left\{\rho = \frac{1}{2} + i\gamma \mid \zeta(\rho) = 0, \quad 0 < \gamma < T\right\}$$

denote the number of zeros *on* the critical line up to height T.

G. H. Hardy (1914) : $N_0(T) \to \infty$ (as $T \to \infty$)

G. H. Hardy-J. E. Littlewood (1921) : $N_0(T) > cT$

A. Selberg (1942) : $N_0(T) > c' N(T)$

N. Levinson (1974) : $N_0(T) > \frac{1}{3} N(T)$

J. B. Conrey (1989) : $N_0(T) > \frac{2}{5} N(T)$

Levinson's method begins with the following fact first proved by Speiser.

Theorem. (Speiser) *RH* $\Longleftrightarrow \zeta'(s) \neq 0$ *in* $0 < \sigma < \frac{1}{2}$.

In the early seventies, N. Levinson and H. L. Montgomery proved a quantitative version of this:

Theorem. (Levinson-Montgomery) $\zeta(s)$ *and* $\zeta'(s)$ *have the same number of zeros inside* $\mathcal{C}$ *up to* $O(\log T)$.

Proof.

$$\Delta \text{ arg } \frac{\zeta'}{\zeta}(s)\bigg|_{\mathcal{C}} = O(\log T),$$

and

$$\Delta \text{ arg } \frac{\zeta'}{\zeta}(s)\bigg|_{\mathcal{C}} = 2\pi(\#\text{ zeros of } \zeta'(s) \text{ in } \mathcal{C} - \#\text{ zeros of } \zeta(s) \text{ in } \mathcal{C})\,.$$

Sketch of Levinson's method

$\zeta'(s)$ has $N'(T)$

zeros here

So does $\zeta(s)$

(up to $O(\log T)$)

and here

Therefore

$$N(T) = N_0(T) + 2N'(T) + O(\log T),$$

or

$$N_0(T) = N(T) - 2N'(T) + O(\log T).$$

We know $N(T)$, so we need an upper bound for $N'(T)$.

$N'(T) =$ the no. of zeros of $\zeta'(s)$ here

$= $ the no. of zeros of $\zeta'(1-s)$ here

By the functional equation for $\zeta(s)$, $\zeta'(s)$ has the same zeros in

$1/2 < \sigma < 2,\ 0 < t < T$ as

$$G(s) = \zeta(s) + \frac{\zeta'(s)}{L(s)}\,,$$

where $L(s) \sim \frac{1}{2\pi} \log s$.

So we need an upper bound for the number of zeros of $G(s)$ in the rectangle on the right.

We apply Littlewood's Lemma to $\mathcal{C}_a$ with $a = \frac{1}{2} - \frac{c}{\log T}$ and $c > 0$.

$$\frac{1}{2\pi}\int_0^T \log|G(a+it)|dt + \mathcal{E} = \sum_{\substack{zeros\ of\ G \\ \in\mathcal{C}_a}} \mathsf{dist}(\rho^*) \geq \sum_{\substack{zeros\ of\ G \\ \in\mathcal{C}_a \\ \beta^*>1/2}} \mathsf{dist}(\rho^*) \geq (1/2-a)N'(T)\,.$$

Thus,

$$
\begin{aligned}
(1/2 - a)N'(T) &\leq \frac{1}{2\pi}\int_0^T \log|GM(a+it)|dt + \mathcal{E} \\
&= \frac{1}{4\pi}\int_0^T \log|GM(a+it)|^2 dt + \mathcal{E} \\
&\leq \frac{T}{4\pi}\log\left(\frac{1}{T}\int_0^T |GM(a+it)|^2 dt\right) + \mathcal{E}\,,
\end{aligned}
$$

where

$$
M(s) = \sum_{n \leq T^\theta} \frac{a_n}{n^s}\,, \qquad a_n = \mu(n) n^{a-1/2}\left(1 - \frac{\log n}{\log T^\theta}\right)\,,
$$

approximates $1/\zeta(s)$.

Thus, we require an estimate for

$$
\int_0^T |GM(a+it)|^2 dt\,.
$$

This is similar to mean values mentioned before.

Levinson (1974) : One can take $\theta = 1/2 - \epsilon$. This gives

$$N_0(T) > \frac{1}{3} N(T) \qquad (\text{as} T \to \infty) \, .$$

J. B. Conrey (1989) : One can take $\theta = 4/7 - \epsilon$. This gives

$$N_0(T) > \frac{2}{5} N(T) \qquad (\text{as} T \to \infty) \, .$$

As a function of θ, the asymptotic estimate for

$$\int_0^T |GM(a + it)|^2 dt \, .$$

is the same in both cases.

D. Farmer has argued that this remains true for θ arbitrarily large.

Farmer's conjecture imples that

$$N_0(T) \sim N(T) \; .$$

VI. Application: The number of simple zeros.

Let

$$N_s(T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0,\, \zeta'(\rho) \neq 0, \quad 0 < \gamma < T\}$$

We believe that for all $T > 0$,

$$N(T) = N_0(T) = N_s(T)\,.$$

H. Montgomery (1974) :

$$RH \Longrightarrow N_s(T) > \frac{2}{3} N(T)$$

via the pair correlation method.

Conrey, Ghosh, Gonek (1999):

$$RH + GLH \Longrightarrow N_s(T) > \frac{19}{27} N(T) = (.703\ldots) N(T)\,.$$

Sketch of the method

This time we use discrete mean values. By the Cauchy–Schwarz inequality

$$\Big|\sum_{0<\gamma<T}\zeta'(1/2+i\gamma)\Big|^2 \le \Big(\sum_{\substack{0<\gamma\le T\\ 1/2+i\gamma\, is\, simple}} 1\Big)\Big(\sum_{0<\gamma<T}|\zeta'(1/2+i\gamma)|^2\Big),$$

An asymptotic estimate for the means provides a lower bound for $N_s(T)$. But this only leads to $N_s(T) > cT$.

We lose in applying the Cauchy–Schwarz inequality. To minimize the loss we mollify $\zeta'(1/2+i\gamma)$ by a Dirichlet polynomial $M_N(s)$.

$$\Big|\sum_{0<\gamma<T}\zeta'(\rho)M_N(\rho)\Big|^2 \le \Big(\sum_{\substack{0<\gamma\le T\\ \rho=1/2+i\gamma\\ is\, simple}} 1\Big)\Big(\sum_{0<\gamma<T}|\zeta'(\rho)M_N(\rho)|^2\Big),$$

An elaboration of the method shows that on the same hypotheses at least 95.5% of the zeros of $\zeta(s)$ are either simple or double.