Zeros of Partial Sums of the Riemann Zeta-Function

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The Riemann zeta-function is that function of the complex variable $s = \sigma + it$, defined in the half-plane $\sigma > 1$ by the absolutely convergent Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which can be continued analytically to a meromorphic function in $\mathbb{C}$ with a simple pole at $s = 1$ with residue 1.

If we decompose the sum above into two parts and estimate the tail of the Dirichlet series trivially, then for $\sigma > 1$ and $X \geq 1$

$$\zeta(s) = \sum_{n=1}^{X} \frac{1}{n^s} + O\left(\frac{X^{1-\sigma}}{\sigma - 1}\right).$$

A crude form of the approximate functional equation extends this approximation into the critical strip, where $0 < \sigma < 1$:

$$\zeta(s) = \sum_{n=1}^{X} \frac{1}{n^s} + \frac{X^{1-s}}{s-1} + O(X^{-\sigma}) \tag{1}$$

uniformly for $\sigma \geq \sigma_0 > 0$ if $X > C|t|/2\pi$, where C is any constant greater than 1.

The second term on the right-hand side above reflects the simple pole of $\zeta(s)$ at $s = 1$, which can be ignored if we are not near it. For example, putting $X = t$ and assuming that $t \geq 1$,

$$\zeta(s) = \sum_{n=1}^{t} \frac{1}{n^s} + O(t^{-\sigma}) \tag{2}$$

uniformly for $\sigma \geq \sigma_0 > 0$.

The Lindelöf Hypothesis is that

$$\zeta(1/2 + it) = O(|t|^{\epsilon}) \quad \text{for every } \epsilon > 0.$$

If this is true, then the length of the partial sums in (1) and (2) can be significantly reduced.

Theorem 1. Let σ be bounded, $|\sigma| \geq 1/2$, and $|s - 1| > 1/10$. Also, let $1 \leq X \leq t^2$. Then a necessary and sufficient condition for the truth of the Lindelöf Hypothesis is that

$$\zeta(s) = \sum_{n=1}^{X} \frac{1}{n^s} + O(X^{1/2-\sigma}|t|^{\epsilon}).$$

Thus, on the Lindelöf Hypothesis, if we stay away from the simple pole of $\zeta(s)$ at $s = 1$, then $\zeta(s)$ is well-approximated by arbitrarily short truncations of its Dirichlet series in the half-plane $\sigma > 1/2$.

However, short truncations of the associated Dirichlet series cannot approximate $\zeta(s)$ well in the left-half of the critical strip, where $0 < \sigma \leq 1/2$. To see this, suppose that the short sum $\sum_{n=1}^{X} 1/n^s$ and $\zeta(s)$ are within ϵ of each other, so that

$$\int_{T}^{2T} \left| \zeta(s) - \sum_{n=1}^{X} \frac{1}{n^s} \right|^2 dt \leq \epsilon^2 T. \tag{3}$$

If we fix $0 < \sigma \leq 1/2$ and take $X < T^{1-\epsilon}$, then

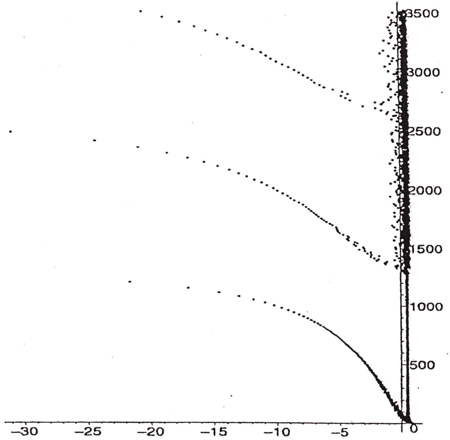
$$\int_{T}^{2T} |\zeta(\sigma + it)|^2 \, dt \sim \begin{cases} C(\sigma) T^{2-2\sigma}, & \text{if } \sigma < 1/2, \\ T \log T, & \text{if } \sigma = 1/2, \end{cases}$$

and

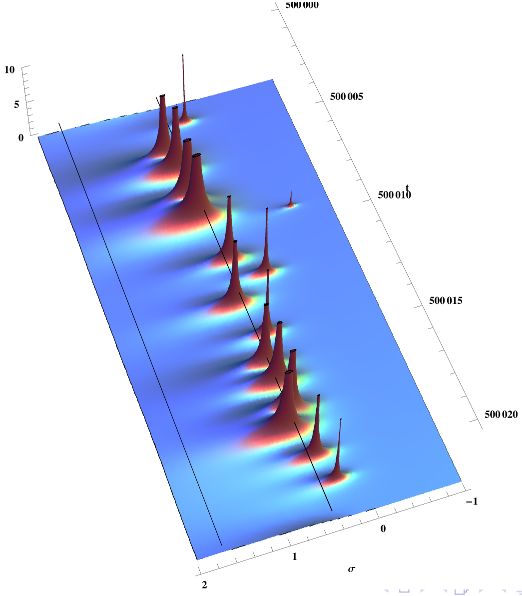
$$\int_{T}^{2T} \left| \sum_{n=1}^{X} \frac{1}{n^s} \right|^2 dt \sim \begin{cases} T \left(\frac{X^{1-2\sigma} - 1}{1 - 2\sigma} \right), & \text{if } \sigma < 1/2, \\ T \log X, & \text{if } \sigma = 1/2. \end{cases}$$

The required contradiction to (3) is obtained by comparing the asymptotics in each of the two cases.

This picture of the first 3,000 suitably normalized zeros of the 211th partial sum of $\zeta(s)$ was generated by Borwein, Fee, Ferguson, and van der Waall for their 2007 paper. It shows a remarkable phenomenon in need of an explanation.



In the surface image, the zeros of the 211th partial sum of $\zeta(s)$, some of which lie off the critical line $\sigma = 1/2$, appear as spikes or peaks. High up the critical strip, the zeros scatter more wildly.



For the notation, let

$$F_X(s) = \sum_{n=1}^{X} \frac{1}{n^s},$$

where $s = \sigma + it$ and $X \geq 2$. Also, let $\rho_X = \beta_X + i\gamma_X$ be a typical zero of $F_X(s)$ and define

$$N_X(T) = \#\{\rho_X = \beta_X + i\gamma_X | F_X(\rho_X) = 0, 0 \leq \gamma_X \leq T\},$$
$$N_X(\sigma, T) = \#\{\rho_X = \beta_X + i\gamma_X | F_X(\rho_X) = 0, 0 \leq \gamma_X \leq T, \beta_X > \sigma\}.$$

If $T = \gamma_X$, define

$$N_X(T) = \lim_{\epsilon \to 0^+} N_X(T + \epsilon),$$
$$N_X(\sigma, T) = \lim_{\epsilon \to 0^+} N_X(\sigma, T + \epsilon).$$

There are two ways to pose questions about $N_X(T)$ and $N_X(\sigma, T)$.

1. Fix X, consider the zeros with $0 \leq \gamma_X \leq T$, and let $T \to \infty$.
2. Ask for uniform results as $X, T \to \infty$.

Theorem 2.

1. (Borwein, Fee, Ferguson, and van der Waall, 2007) The zeros of $F_X(s)$ lie in the strip $\alpha < \sigma < \beta$, where α and β are the unique solutions of the algebraic equations

$$1 + 2^{-\sigma} + \cdots + (X-1)^{-\sigma} = X^{-\sigma},$$
$$2^{-\sigma} + 3^{-\sigma} + \cdots + X^{-\sigma} = 1,$$

respectively. In particular, $\alpha > -X$ and $\beta < 1.72865$.

2. (Turán, 1948) For X large enough, $F_X(s)$ is nonzero in the half-plane

$$\sigma \geq 1 + 2\frac{\log\log X}{\log X}.$$

3. (Montgomery, 1983) For any constant $c > 4/\pi - 1$ there exists a number $X_0 = X_0(c)$ such that $F_X(s)$ has at most a finite number of zeros in the half-plane

$$\sigma > 1 + c\frac{\log\log X}{\log X} \quad \text{whenever } X \geq X_0.$$

Theorem 3. Let $X = X(T)$ such that $X \to \infty$ as $T \to \infty$. Then for T large enough, we have

$$N_X(T) = \frac{T}{2\pi} \log X + O(X).$$

Proof. First, show that $F_X(s) \neq 0$ in the half-plane $\Re s \geq 2$. Write

$$\begin{aligned} F_X(\sigma + it) &= \sum_{n=1}^{X} n^{\sigma} e^{-it \log n} = \sum_{n=1}^{X} \frac{\cos(t \log n) - i \sin(t \log n)}{n^{\sigma}} \\ &= \Re F_X(\sigma + it) + i \Im F_X(\sigma + it). \end{aligned}$$

If $\sigma \geq 2$, then

$$\Re F_X(\sigma + it) \geq 1 - \sum_{n=2}^{X} \frac{1}{n^2} > 2 - \sum_{n=1}^{\infty} \frac{1}{n^2} = 2 - \frac{\pi^2}{6} > \frac{1}{3}.$$

So let $\mathcal{C}$ be the rectangle with vertices $-U$, 2, $2+iT$, and $-U+iT$, where $U \geq X$, described in the positive sense. Without any loss of generality, assume that $F_X(s) \neq 0$ along the top edge of $\mathcal{C}$.

By the principle of the argument,

$$N_X(T) = \frac{1}{2\pi} \triangle_{\mathcal{C}} \arg F_X(s),$$

where $\triangle_{\mathcal{C}}$ denotes the change in $\arg F_X(s)$ as s traverses $\mathcal{C}$ in the positive sense.

1. If $t \neq \gamma_X$, define $\arg F_X(\sigma + it)$ as the value obtained by continuous variation along the line from 2 to $2 + it$ and then to $\sigma + it$, starting with the value 0.
2. If $t = \gamma_X$, simply define

$$\arg F_X(\sigma + it) = \lim_{\epsilon \to 0^+} \arg F_X(\sigma + i(t + \epsilon)).$$

Along the bottom edge of $\mathcal{C}$, $t = 0$ and $-U \le \sigma \le 2$. We have

$$F_X(\sigma) = \sum_{n=1}^{X} \frac{1}{n^\sigma} > 0,$$

so that

$$\triangle \arg F_X(\sigma) \Big|_{-U}^{2} = 0.$$

Along the right edge of $\mathcal{C}$, $\sigma = 2$ and $0 \le t \le T$. Since

$$\Re F_X(2 + it) > 0,$$

we have

$$\triangle \arg F_X(2 + it) \Big|_{0}^{T} = O(1).$$

Along the top edge of $\mathcal{C}$, $t = T$ and $-U \le \sigma \le 2$. We apply Descartes' Rule of Signs to $\Im F_X(\sigma + iT)$.

Theorem 4. (Descartes' Rule of Signs) Let $a_1, a_2, \ldots, a_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be real constants such that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Denote by Z the number of real zeros of the entire function

$$G(x) = a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + \cdots + a_n e^{\lambda_n x}$$

and by C the number of changes of signs in the sequence $a_1, a_2, \ldots, a_n$. Then $C - Z \geq 0$ and even.

As a result, the number of zeros of

$$\Im F_X(\sigma + iT) = -\sum_{n=1}^{X} \frac{\sin(T \log n)}{n^\sigma}$$

throughout $-U \leq \sigma \leq 2$ is

$$\begin{aligned} &\leq \text{The number of changes in signs in the sequence} \\ &\qquad \sin(T\log 2), \sin(T\log 3), \ldots, \sin(T\log X) \\ &= O(X). \end{aligned}$$

Moreover,

$$|\triangle \arg F_X(\sigma + iT)| < \pi$$

between consecutive zeros of $\Im F_X(s)$. Thus

$$\triangle \arg F_X(\sigma + iT) \Big|_2^{-U} = O(X).$$

Along the left edge of $\mathcal{C}$, $\sigma = -U$ and $0 \leq t \leq T$. Note that

$$F_X(-U + it) = \sum_{n=1}^{X-1} n^{U-it} + X^{U-it},$$

where

$$\left|\sum_{n=1}^{X-1} n^{U-it}\right| \leq \sum_{n=1}^{X-1} n^U \leq \int_1^X w^U \, dw < \frac{X^{U+1}}{U} \leq \frac{X^{U+1}}{X} = X^U,$$

if $U \geq X$.

Thus, if $U \geq X$

$$\begin{aligned}\triangle \arg F_X(-U+it)\Big|_T^0 &= \triangle \arg\left(1+2^{U-it}+\cdots+(X-1)^{U-it}\right)\Big|_T^0 \\ &\quad + \triangle \arg X^{U-it}\Big|_T^0 . \\ &= \triangle \arg\left(-t\log X\right)\Big|_T^0 + O(1) \\ &= T\log X + O(1).\end{aligned}$$

Combining all estimates we obtain the required result.

Next, we estimated $N_X(\sigma, T)$ for $\sigma \geq 1/2$.

Theorem 5. If $X = X(T)$ such that $X \to \infty$ as $T \to \infty$ and if $X = O(T)$, then

$$N_X(\sigma, T) = O(TX^{-2(\sigma-1/2)}\log^5 T + \log^2 T)$$

uniformly for $1 \leq \sigma \leq 2$. Also, if $X = o(T)$, then

$$N_X(\sigma, T) = O(T(\min(X, T/X))^{-2(\sigma-1/2)}\log^5 T)$$

uniformly for $1/2 \leq \sigma \leq 1$.

Outline of the proof. Define

$$f(s) = F_X(s)M_Y(s) - 1 = \sum_{n=1}^{XY} \frac{a(n)}{n^s},$$

where

$$M_Y(s) = \sum_{n=1}^{Y} \frac{\mu(n)}{n^s},$$

$\mu(n)$ is the Möbius function and $Y = O(T)$. We have $a(1) = 0$ and

$$a(n) = \sum_{\substack{d|n \\ d \le Y \\ n/d \le X}} \mu(d) \quad \text{for } 1 < n \le XY.$$

Moreover, $a(n) = 0$ for $1 \le n \le \min(X, Y)$. Therefore

$$f(s) = \sum_{Z < n \le XY} \frac{a(n)}{n^s},$$

where $Z = \min(X, Y)$. (If $\sigma \ge 2$, we will get $|f(s)|^2 < 1/2$.)

Now set

$$h(s) = 1 - f^2(s) = F_X(s)M_Y(s)(2 - F_X(s)M_Y(s)),$$

so that

1. $h(s)$ is holomorphic and vanishes at the zeros of $F_X(s)$.
2. $h(s) \neq 0$ for $\sigma \geq 2$ and Z large enough, since $|f(s)|^2 < 1/2$.

We now apply a lemma of Littlewood to $h(s)$, which is:

Theorem 6. (Littlewood's lemma) Let $g(s)$ be analytic and nonzero on the rectangle $\mathcal{R}$ with vertices σ_0, σ_1, $\sigma_1 + iT$, and $\sigma_0 + iT$, where $\sigma_0 < \sigma_1$. Then

$$2\pi \sum_{\rho \in \mathcal{R}} \mathrm{Dist}(\rho) = \int_0^T \log|g(\sigma_0 + it)|\, dt - \int_0^T \log|g(\sigma_1 + it)|\, dt$$
$$\int_{\sigma_0}^{\sigma_1} \arg g(\sigma + iT)\, d\sigma - \int_{\sigma_0}^{\sigma_1} \arg g(\sigma)\, d\sigma,$$

where the sum is taken over the zeros ρ of $\mathcal{R}$ and $\mathrm{Dist}(\rho)$ is the distance from ρ to the left edge of $\mathcal{R}$.

As a result,

$$2\pi \sum_{\substack{0\le\gamma_X\le T\\ \beta_X>\sigma_0}} (\beta_X - \sigma_0) = \int_0^T \log|h(\sigma_0 + it)|\, dt - \int_0^T \log|h(\sigma_1 + it)|\, dt$$

$$+ \int_{\sigma_0}^{\sigma_1} \arg h(\sigma + iT)\, d\sigma - \int_{\sigma_0}^{\sigma_1} \arg h(\sigma)\, d\sigma,$$

where

$$\frac{1}{2} - \frac{1}{\log T} \le \sigma_0 \le 2.$$

To treat these integrals we employ two well-known theorems.

Theorem 7. (Montgomery-Vaughan, 1974)

$$\int_0^T \left| \sum_{n=1}^{N} \frac{a(n)}{n^s} \right|^2 dt = \sum_{n=1}^{N} \frac{|a(n)|^2}{n^{2\sigma}} (T + O(n)).$$

Lemma 1. (See Titchmarsh, Section 9.4.) Let $0 \le \alpha < \beta < 2$. Let $f(s)$ be an analytic function, real for real s, regular for $\sigma \ge \alpha$ except at $s = 1$. Also, let

$$|\Re f(2+it)| \ge m > 0,$$
$$|f(\sigma' + it')| \le M(\sigma, t) \quad (\sigma' > \sigma, 1 \le t' \le t).$$

Then if T is not the ordinate of a zero of $f(s)$

$$|\arg f(\sigma + iT)| \le \frac{\pi}{\log\{(2-\alpha)/(2-\beta)\}}\left(\log M(\sigma, T+2) + \log\frac{1}{m}\right) + \frac{3\pi}{2}$$

for $\sigma \ge \beta$.

We work to find that

$$\sum_{\substack{0 \le \gamma_X \le T \\ \beta_X > \sigma_0}} (\beta_X - \sigma_0) = O((TZ^{1-2\sigma_0} + (XY)^{2-2\sigma_0}) \log^4 T + \log T).$$

We use this with various choices for Z to derive our theorem.

Corollary 1. If $X = X(T)$ such that $X \to \infty$ as $T \to \infty$ and if $X = o(T)$, then for any constant $c_1 \geq 5/2$ and T large enough

$$\beta_X \leq \frac{1}{2} + \frac{c_1 \log\log T}{\log(\min(X, T/X))}$$

for almost all zeros of $F_X(s)$ with $0 \leq \gamma_X \leq T$.

We also proved a conditional result along the same line.

Theorem 8. Assume the Riemann Hypothesis. If $X = X(T)$ such that $X \to \infty$ as $T \to \infty$ and if $X < T^2$, then there is an absolute constant c_2 such that, for T large enough,

$$\beta_X \leq \frac{1}{2} + \frac{c_2 \log T}{\log X \cdot \log\log T}$$

for all zeros of $F_X(s)$ with $\sqrt{T} < \gamma_X \leq T$.

Next, we showed that the zeros to the right of the critical line $\sigma = 1/2$ are, on average, close to it.

Theorem 9. If $X = X(T)$ such that $X \to \infty$ as $T \to \infty$ and if $X = O(T)$, then

$$\sum_{\substack{0 \le \gamma_X \le T \\ \beta_X > 1/2}} (\beta_X - 1/2) \le \frac{T}{4\pi} \log\log X + O(T).$$

Finally, we obtained information about the zeros for arbitrary values of $\sigma < 1/2$.

Theorem 10. If $X = X(T)$ such that $X \to \infty$ as $T \to \infty$ and if $X = o(T)$, then

$$\sum_{\substack{0 \le \gamma_X \le T \\ \beta_X > \sigma}} (\beta_X - \sigma) \le (1/2 - \sigma) \frac{T}{2\pi} \log X - \frac{T}{4\pi} \log(1/2 - \sigma)$$

$$+ O((1 + |\sigma|)X) + O(T).$$

Open Question. An important question we left unanswered is whether or not

$$\sum_{\substack{0 \le \gamma_X \le T \\ \beta_X > \sigma}} (\beta_X - \sigma) \sim (1/2 - \sigma) \frac{T}{2\pi} \log X$$

when σ is bounded and less than $1/2$. An answer to this would require an asymptotic estimate for

$$\int_0^T \log|F_X(\sigma_0 + it)|\, dt$$

rather than an upper bound.

Thank you.