Pair Correlation of the

Zeros of the Riemann Zeta-Function

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1. An Explicit Formula

Let $s = \sigma + it$. For $\sigma > 1$

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}.$$

For $c > 0$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^w}{w}dw = \begin{cases} 1 & \text{if} \quad y > 1, \\ \frac{1}{2} & \text{if} \quad y = 1, \\ 0 & \text{if} \quad y \le 1. \end{cases}$$

Thus, if $x > 1, x \neq p^k$ and $\sigma + c > 1$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta'}{\zeta}(s+w)\frac{x^w}{w}dw = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{(x/n)^w}{w}dw\right)$$
$$= -\sum_{n \le x} \frac{\Lambda(n)}{n^s}.$$

Pulling the contour left to $\operatorname{Re} s = -\infty$ gives

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta'}{\zeta}(s+w)\frac{x^w}{w}dw = \sum_{\rho} \frac{x^{\rho-s}}{\rho - s} + \frac{x^{1-s}}{s-1} - \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s} + \frac{\zeta'}{\zeta}(s).$$

Here $s \neq 1, \rho$ (a nontrivial zero), or $-2n$.

Equate these.

Explicit Formula.

$$\sum_{n\le x}\frac{\Lambda(n)}{n^s}=-\frac{\zeta'}{\zeta}(s)+\frac{x^{1-s}}{1-s}-\sum_{\rho}\frac{x^{\rho-s}}{\rho-s}+\sum_{n=1}^{\infty}\frac{x^{-2n-s}}{2n+s}$$

for $s\neq 1,\rho,-2n$ and $x>1, x\neq p^k$.

Note that case $s=0$:

$$\psi(x)=\sum_{n\le x}\Lambda(n)=-\frac{\zeta'}{\zeta}(0)+x-\sum_{\rho}\frac{x^{\rho}}{\rho}+\sum_{n=1}^{\infty}\frac{x^{-2n}}{2n}.$$

Rewrite the explicit formula as

$$\sum_{\rho}\frac{x^{\rho}}{\rho-s}=-x^s\left(\frac{\zeta'}{\zeta}(s)+\sum_{n\le x}\frac{\Lambda(n)}{n^s}\right)+\frac{x}{1-s}+\sum_{n=1}^{\infty}\frac{x^{-2n}}{2n+s}$$

Assume RH. Then $\rho=\frac{1}{2}+i\gamma$. If we take $s=\frac{3}{2}+it$ we obtain

$$-x^{\frac{1}{2}}\sum_{\gamma}\frac{x^{i\gamma}}{1+i(t-\gamma)}=x^{\frac{3}{2}+it}\sum_{n>x}\frac{\Lambda(n)}{n^{3/2+it}}-\frac{x}{\frac{1}{2}+it}+\sum_{n=1}^{\infty}\frac{x^{-2n}}{2n+\frac{3}{2}+it}$$

Taking $s=-\frac{1}{2}+it$ gives

$$x^{\frac{1}{2}}\sum_{\gamma}\frac{x^{i\gamma}}{1-i(t-\gamma)}=-\,x^{-\frac{1}{2}+it}\sum_{n\le x}\frac{\Lambda(n)}{n^{-1/2+it}}-x^{-\frac{1}{2}+it}\frac{\zeta'}{\zeta}(-\tfrac{1}{2}+it)$$

$$+\frac{x}{\frac{3}{2}-it}+\sum_{n=1}^{\infty}\frac{x^{-2n}}{2n-\frac{1}{2}+it}$$

We replace the $\zeta'/\zeta(-\frac{1}{2}+it)$ term here by $-\log(|t|+2)+O(1)$.

This follows from

$$\zeta'/\zeta(s) = \chi'/\chi(s) - \zeta'/\zeta(1-s),$$

so that

$$\zeta'/\zeta(-\tfrac{1}{2}+it) = \chi'/\chi(-\tfrac{1}{2}+it) - \zeta'/\zeta(\tfrac{3}{2}-it) = -\log(|t|+2) + O(1).$$

The last formula becomes

$$x^{\frac{1}{2}} \sum_{\gamma} \frac{x^{i\gamma}}{1-i(t-\gamma)} = -\, x^{-\frac{1}{2}+it} \sum_{n \le x} \frac{\Lambda(n)}{n^{-1/2+it}} + \frac{x}{\frac{3}{2}-it}$$
$$+ x^{-\frac{1}{2}+it}(\log(|t|+2) + O(1)) + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n-\frac{1}{2}+it}.$$

Subtract this from the first, which was

$$-x^{\frac{1}{2}} \sum_{\gamma} \frac{x^{i\gamma}}{1+i(t-\gamma)} = x^{\frac{3}{2}+it} \sum_{n>x} \frac{\Lambda(n)}{n^{3/2+it}} - \frac{x}{\frac{1}{2}+it} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n+\frac{3}{2}+it}.$$

The difference on the left-hand side is

$$-2x^{\frac{1}{2}} \sum_{\gamma} \frac{x^{i\gamma}}{1+(t-\gamma)^2}.$$

The difference on the right-hand side is

$$x^{it} \sum_{n \le x} \frac{\Lambda(n)}{n^{it}} \left(\frac{n}{x}\right)^{\frac{1}{2}} + x^{it} \sum_{n>x} \frac{\Lambda(n)}{n^{it}} \left(\frac{x}{n}\right)^{\frac{3}{2}} + \frac{2x}{(\frac{1}{2}+it)(-\frac{3}{2}+it)}$$
$$+ x^{-\frac{1}{2}+it}(-\log(|t|+2) + O(1)) - 2\sum_{n=1}^{\infty} \frac{x^{-2n}}{(2n+\frac{3}{2}+it)(2n-\frac{1}{2}+it)}.$$

Both the LHS and RHS are continuous in x, so we need no longer exclude $x = 1$ or $x = p^k$.

Also, the final sum is $\ll x^{-2}/(|t| + 2)$.

Thus, equating the two expressions and using the notation

$$a_x(n) = \min\left(\left(\frac{n}{x}\right)^{\frac{1}{2}}, \left(\frac{x}{n}\right)^{\frac{3}{2}}\right)$$

we have the following theorem.

Theorem. *Assume RH. If $x \geq 1$, then*

$$-2x^{\frac{1}{2}} \sum_{\gamma} \frac{x^{i\gamma}}{1 + (t-\gamma)^2}$$
$$= x^{it} \sum_{n=2}^{\infty} \frac{\Lambda(n) a_x(n)}{n^{it}} + \frac{2x}{(\frac{1}{2} + it)(-\frac{3}{2} + it)}$$
$$+ x^{-\frac{1}{2}+it}\big(-\log(|t| + 2) + O(1)\big) + O\Big(\frac{x^{-2}}{|t| + 2}\Big).$$

2. Montgomery's Theorem

We rewrite

$$
\begin{aligned}
-2x^{\frac{1}{2}} \sum_{\gamma} & \frac{x^{i\gamma}}{1+(t-\gamma)^2} \\
&= x^{it} \sum_{n=2}^{\infty} \frac{\Lambda(n) a_x(n)}{n^{it}} + \frac{2x}{(\frac{1}{2}+it)(-\frac{3}{2}+it)} \\
&\quad + x^{-\frac{1}{2}+it}\big(-\log(|t|+2)+O(1)\big) + O\Big(\frac{x^{-2}}{|t|+2}\Big)
\end{aligned}
$$

as

$$
L(x,t) = R(x,t)
$$

Montgomery's pair correlation theorem is proved by calculating both sides of $\int_0^T |L(x,t)|^2 dt = \int_0^T |R(x,t)|^2 dt$. We carry this out now skipping only a few minor details.

Calculation of $\int_0^T |L(x,t)|^2 dt$.

We have

$$
\int_0^T |L(x,t)|^2 dt = 4x \int_0^T \Big| \sum_{\gamma} \frac{x^{i\gamma}}{1+(t-\gamma)^2} \Big|^2 dt
$$

It is not difficult to show that one can truncate the sum over γ's to $\sum_{0<\gamma\le T}$ and extend the integration to $\int_{-\infty}^{\infty}$ at a cost of $O(\log^3 T)$.

Assuming this, we have

$$\int_0^T |L(x,t)|^2 dt = 4x \int_{-\infty}^{\infty} \Big| \sum_{0<\gamma\le T} \frac{x^{i\gamma}}{1+(t-\gamma)^2} \Big|^2 dt + O(x\log^3 T).$$

Squaring out and integrating, we find that this equals

$$4x \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')} \int_{-\infty}^{\infty} \frac{dt}{(1+(t-\gamma)^2)(1+(t-\gamma')^2)} + O(x\log^3 T)$$

$$=2\pi x \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')} \frac{4}{4+(\gamma-\gamma')^2} + O(x\log^3 T)$$

$$=2\pi x \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')} w(\gamma-\gamma') + O(x\log^3 T)$$

$$=2\pi x F(x,T) + O(x\log^3 T),$$

say.

Exercise. Show that

$$\int_{-\infty}^{\infty} \frac{dt}{(1+(t-\gamma)^2)(1+(t-\gamma')^2)} = \frac{2\pi}{4+(\gamma-\gamma')^2}.$$

Hint: write $1+(t-\gamma)^2 = (t-(\gamma+i))(t-(\gamma-i))$, etc. and use the calculus of residues.

Note that

$$F(x,T) = \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')} w(\gamma-\gamma') = \frac{2}{\pi} \int_{-\infty}^{\infty} \Big| \sum_{0<\gamma\le T} \frac{x^{i\gamma}}{1+(t-\gamma)^2} \Big|^2 dt.$$

From

$$F(x,T) = \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')} w(\gamma-\gamma') = \frac{2}{\pi}\int_{-\infty}^{\infty} \Big| \sum_{0<\gamma\le T} \frac{x^{i\gamma}}{1+(t-\gamma)^2} \Big|^2 dt.$$

We have

$$F(x,T) \ge 0 \qquad \text{and} \qquad F(1/x,T) = F(x,T) \quad (x>0).$$

We have now shown that

$$\int_0^T |L(x,t)|^2 dt = 2\pi x F(x,T) + O(x\log^3 T).$$

Calculation of $\int_0^T |R(x,t)|^2 dt$.

The right-hand side of the explicit formula is

$$\begin{aligned} R(x,t) =& x^{it} \sum_{n\le x} \frac{\Lambda(n)a_x(n)}{n^{it}} + \frac{2x}{(\frac{1}{2}+it)(-\frac{3}{2}+it)} \\ &+ x^{-\frac{1}{2}+it}\big(-\log(|t|+2) + O(1)\big) + O\Big(\frac{x^{-2}}{|t|+2}\Big). \end{aligned}$$

First we calculate the mean square of each term on the right.

By the Montgomery-Vaughan mean value theorem for Dirichlet series

$$\begin{aligned}\int_0^T \left| x^{it} \sum_{n=2}^{\infty} \frac{\Lambda(n)a_x(n)}{n^{it}} \right|^2 dt \\ &= \sum_{n=2}^{\infty} \Lambda^2(n)a_x^2(n)(T+O(n)) \\ &= \frac{1}{x}\sum_{n\le x} \Lambda^2(n)n(T+O(n)) + x^3 \sum_{n>x} \frac{\Lambda^2(n)}{n^3}(T+O(n)).\end{aligned}$$

By the prime number theorem this equals

$$\begin{aligned}&\frac{1}{x}\Big(T\frac{x^2}{2}\log x + O(x^3\log x)\Big) + x^3\Big(T\frac{1}{2x^2}\log x + O\Big(\frac{1}{x}\log x\Big)\Big) \\ &= xT(\log x + O(1)) + O(x^2\log x).\end{aligned}$$

Secondly,

$$\int_0^T \left|\frac{2x}{(\frac{1}{2}+it)(-\frac{3}{2}+it)}\right|^2 dt \ll x^2.$$

Thirdly,

$$\int_0^T \left|x^{-\frac{1}{2}+it}\big(\log(t+2)+O(1)\big)\right|^2 dt = \frac{1}{x}(T\log^2 T + O(\log T)).$$

And finally,

$$\int_0^T \left|\frac{x^{-2}}{t+2}\right|^2 dt \ll x^{-4}.$$

We use these as follows.

$R(x,t)$, the right-hand side of our explicit formula, is a sum of 4 terms:

$$\begin{aligned} R(x,t) =& x^{it}\sum_{n\le x}\frac{\Lambda(n)a_x(n)}{n^{it}}+\frac{2x}{(\frac{1}{2}+it)(-\frac{3}{2}+it)} \\ &+x^{-\frac{1}{2}+it}\big(-\log(|t|+2)+O(1)\big)+O\Big(\frac{x^{-2}}{|t|+2}\Big). \end{aligned}$$

Write this as

$$R(x,t)=A_1(x,t)+A_2(x,t)+A_3(x,t)+A_4(x,t)$$

For a given x we let

$$M_i=\int_0^T|A_i(x,t)|^2dt,$$

but ordered so that

$$M_1\ge M_2\ge M_3\ge M_4.$$

It is easy to show by the Cauchy-Schwarz inequality that

$$\begin{aligned} \int_0^T|R(x,t)|^2dt=&\int_0^T|A_1(t)+A_2(t)+A_3(t)+A_4(t)|^2dt \\ =&M_1+O((M_1M_2)^{\frac{1}{2}}). \end{aligned}$$

Exercise. Show this.

For $1 \leq x \leq \log^{\frac{3}{4}} T,\ M_1$ is given by

$$\int_0^T \left| x^{-\frac{1}{2}+it}\big(\log(t+2) + O(1)\big)\right|^2 dt = \frac{1}{x}(T\log^2 T + O(\log T))$$

and M_2 by

$$\int_0^T \left| x^{it} \sum_{n=2}^{\infty} \frac{\Lambda(n) a_x(n)}{n^{it}} \right|^2 dt = xT(\log x + O(1)) + O(x^2 \log x).$$

For $\log^{\frac{3}{4}} T \leq x \leq \log^{\frac{3}{2}} T$, all terms are $o(xT\log T)$.

For $\log^{\frac{3}{2}} T \leq x \leq o(T), \quad M_1$ is given by

$$\int_0^T \left| x^{it} \sum_{n=2}^{\infty} \frac{\Lambda(n) a_x(n)}{n^{it}} \right|^2 dt = xT(\log x + O(1)) + O(x^2 \log x)$$

and M_2 is given by

$$\int_0^T \left| x^{-\frac{1}{2}+it}\big(\log(t+2) + O(1)\big)\right|^2 dt = \frac{1}{x}(T\log^2 T + O(\log T)).$$

It follows that

$$\int_0^T |R(x,t)|^2 dt = xT(\log x + o(\log T)) + O(x^2 \log x) + \frac{T}{x}\log^2 T(1 + o(1)).$$

Recall that

$$\int_0^T |L(x,t)|^2 dt = 2\pi x F(x,T) + O(x\log^3 T).$$

Thus

$$F(x,T) = \frac{T}{2\pi}(\log x + o(\log T)) + O(x \log x) + \frac{T}{2\pi x^2} \log^2 T(1+o(1)).$$

Set $x = T^\alpha$ and

$$F(\alpha) = F(\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} F(T^\alpha, T).$$

Then we have proved the

Theorem. (Montgomery's Theorem) *Assume RH. Let*

$$F(\alpha) = F(\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\leq T} w(\gamma - \gamma')\, T^{i\alpha(\gamma-\gamma')}$$

where γ and γ' run over ordinates of zeros of the Riemann zeta-function and $w(u) = 4/(4+u^2)$. Then for $\alpha \in \mathbb{R}$, $F(\alpha)$ is real, even, and nonnegative. Moreover, for any $\epsilon > 0$

$$F(\alpha) = (1 + o(1))T^{-2\alpha} \log T + \alpha + o(1)$$

uniformly for $|\alpha| \leq 1 - \epsilon$ as $T \to \infty$.

It was later shown that the formula in fact holds for $|\alpha| \leq 1$.

3. Applications

The way one retrieves information from Montgomery's Theorem is as follows. Let

$$\hat{r}(\alpha) = \int_{-\infty}^{\infty} r(u) e^{-2\pi i \alpha u}\, du$$

be the Fourier transform of r, and let

$$r(u) = \int_{-\infty}^{\infty} \hat{r}(\alpha) e^{2\pi i \alpha u}\, d\alpha$$

be the inverse transform. Then

$$\begin{aligned}
\Big(\frac{T}{2\pi}\log T\Big)\int_{-\infty}^{\infty} F(\alpha)\hat{r}(\alpha)\, d\alpha &= \int_{-\infty}^{\infty} \sum_{0<\gamma,\gamma'\le T} w(\gamma-\gamma')\; T^{i\alpha(\gamma-\gamma')}\hat{r}(\alpha)\, d\alpha \\
&= \sum_{0<\gamma,\gamma'\le T} w(\gamma-\gamma') \int_{-\infty}^{\infty} T^{i\alpha(\gamma-\gamma')}\hat{r}(\alpha)\, d\alpha \\
&= \sum_{0<\gamma,\gamma'\le T} w(\gamma-\gamma') r\Big((\gamma-\gamma')\frac{\log T}{2\pi}\Big).
\end{aligned}$$

Thus, the integral of $F(\alpha)$ against a kernel $\hat{r}$ produces a sum involving the inverse transform r evaluated at the differences of pairs of ordinates.

Since Montgomery's Theorem is only valid in the range $-1 < \alpha < 1$, we only use kernels $\hat{r}(\alpha)$ supported on $(-1, 1)$.

Application to Counting Simple Zeros.

Consider the Fourier transform pair

$$r(u) = \Big(\frac{\sin \pi\lambda u}{\pi\lambda u}\Big)^2, \qquad \hat{r}(\alpha) = \frac{1}{\lambda}\max\Big(1 - \frac{|\alpha|}{\lambda}, 0\Big) \qquad (\lambda > 0).$$

We use this pair in

$$\sum_{0<\gamma,\gamma'\le T} r\Big((\gamma-\gamma')\frac{\log T}{2\pi}\Big)w(\gamma-\gamma') = \Big(\frac{T}{2\pi}\log T\Big)\int_{-\infty}^{\infty} F(\alpha)\hat{r}(\alpha)\,d\alpha$$

and evaluate the RHS using Montgomery's Theorem.

We need the support of $F(\alpha)$ to be in $(-1, 1)$, so we take $\lambda < 1$.

We find that

$$\begin{aligned}
\sum_{0<\gamma,\gamma'\le T} & \Big(\frac{\sin((\lambda/2)(\gamma-\gamma')\log T)}{(\lambda/2)(\gamma-\gamma')\log T}\Big)^2 w(\gamma-\gamma') \\
&= \Big(\frac{T}{2\pi}\log T\Big)\frac{1}{\lambda}\int_{-\infty}^{\infty} F(\alpha)\max\Big(1-\frac{|\alpha|}{\lambda}, 0\Big)\,d\alpha \\
&= \Big(\frac{T}{2\pi}\log T\Big)\frac{1}{\lambda}\int_{-\lambda}^{\lambda} F(\alpha)\Big(1-\frac{|\alpha|}{\lambda}\Big)\,d\alpha \\
&\sim \Big(\frac{T}{2\pi}\log T\Big)\frac{2}{\lambda}\int_{0}^{\lambda} \Big(\alpha + T^{-2\alpha}\log T\Big)\Big(1-\frac{\alpha}{\lambda}\Big)\,d\alpha \\
&\sim \Big(\frac{1}{\lambda}+\frac{\lambda}{3}\Big)\frac{T}{2\pi}\log T\,.
\end{aligned}$$

Montgomery used this to obtain a lower bound for the number of simple zeros of the zeta-function as follows.

Observe that if $\rho = \frac{1}{2} + i\gamma$ is a zero of multiplicity $m(\rho)$, then

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ \gamma=\gamma'}} 1 = \sum_{0<\gamma\leq T} m(\rho);$$

on each side, γ's are counted according to their multiplicities.

Clearly

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ \gamma=\gamma'}} 1 \leq \sum_{0<\gamma,\gamma'\leq T} \left(\frac{\sin((\lambda/2)(\gamma-\gamma')\log T)}{(\lambda/2)(\gamma-\gamma')\log T}\right)^2 w(\gamma-\gamma')\,.$$

We saw that the RHS is $\sim \left(\frac{1}{\lambda}+\frac{\lambda}{3}\right)\frac{T}{2\pi}\log T$. Take $\lambda = 1-\epsilon$ in this to obtain

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ \gamma=\gamma'}} 1 \leq \left(\frac{4}{3}+o(1)\right)\frac{T}{2\pi}\log T\,.$$

Replacing the LHS by $\sum_{0<\gamma\leq T} m(\rho)$, we find that

$$\sum_{0<\gamma\leq T} m(\rho) \leq \left(\frac{4}{3}+o(1)\right)\frac{T}{2\pi}\log T\,.$$

Finally, we see that

$$\sum_{\substack{0<\gamma\le T\\ \frac12+i\gamma\text{ is simple}}} 1 \ge \sum_{0<\gamma\le T}(2-m_\rho)$$
$$\ge \Big(2-\frac{4}{3}+o(1)\Big)\frac{T}{2\pi}\log T$$
$$\ge \Big(\frac{2}{3}+o(1)\Big)\frac{T}{2\pi}\log T.$$

Thus we have the

Theorem. (Montgomery) *Assume RH. Let $N_s(T)$ denote the number of simple zeros of $\zeta(s)$ with ordinates in $(0,T]$. Then*

$$N_s(T) \ge \Big(\frac{2}{3}+o(1)\Big)N(T).$$

Recall that in Lecture II we outlined a proof that

$$N_s(T) \ge (19/27+o(1))N(T).$$

Note that $19/27 = .7037... > .666... = 2/3$, so that result was stronger. However, so were the hypotheses, for there we needed to assume the Generalized Lindeloff Hypothesis as well as RH.

Montgomery's Conjecture.

We determined $F(\alpha)$ by calculating the mean square of both sides of

$$-2x^{\frac{1}{2}}\sum_{\gamma}\frac{x^{i\gamma}}{1+(t-\gamma)^2}$$

$$(1) \qquad = x^{it}\sum_{n=2}^{\infty}\frac{\Lambda(n)a_x(n)}{n^{it}}+\frac{2x}{(\frac{1}{2}+it)(-\frac{3}{2}+it)}$$

$$+\,x^{-\frac{1}{2}+it}\big(-\log(|t|+2)+O(1)\big)+O\Big(\frac{x^{-2}}{|t|+2}\Big).$$

and setting $x=T^{\alpha}$.

The restriction $0\le\alpha<1$ (corresponding to $1\le x=o(T)$) arose because we used the Montgomery-Vaughan mean value theorem to calculate the mean square of the Dirichlet series.

This only required estimates for "diagonal" terms involving $\sum_{n\le y}\Lambda^2(n)$, and is satisfactory when $x=o(T)$.

If $\alpha\ge 1$, then $x\ge T$, and "off-diagonal" terms contribute to the mean square. These require estimates for the sums $\sum_{n\le y}\Lambda(n)\Lambda(n+h)$ uniform in h.

Montgomery used a strong form of the Hardy–Littlewood twin prime conjecture to estimate these and arrived at the

Conjecture. (Montgomery's Conjecture) *For any fixed A we have*

$$F(\alpha, T) = 1 + o(1)$$

uniformly for $1 \le \alpha \le A$ as $T \to \infty$.

This and Montgomery's Theorem determine $F(\alpha)$ for all α.

One can use the conjecture to integrate $F(\alpha)$ against a much wider class of kernels than just those supported in $(-1, 1)$.

Using an appropriate kernel (a characteristic function) Montgomery obtained

Conjecture. (Pair Correlation Conjecture) *For any fixed $\beta > 0$, we have*

$$\sum_{\substack{0<\gamma,\gamma'\le T \\ 0<\gamma'-\gamma\le 2\pi\beta/\log T}} 1 \sim \left(\int_0^\beta 1 - \left(\frac{\sin \pi x}{\pi x} \right)^2 dx \right) \frac{T}{2\pi} \log T$$

as T tends to infinity.

An enormous amount of data concerning the zeros has been collected and analyzed by A. M. Odlyzko [O], and the fit with the conjecture is remarkable.

As an example of the type of information we can deduce from

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ 0<\gamma'-\gamma\leq 2\pi\beta/\log T}} 1 \sim \left(\int_0^\beta 1-\left(\frac{\sin \pi x}{\pi x}\right)^2 dx\right)\frac{T}{2\pi}\log T\,,$$

note that this implies that infinitely many zeros must have another zero no farther away than $2\pi\beta/\log\gamma$, no matter how small β is.

Hence

$$\liminf_{n\to\infty}(\gamma_{n+1}-\gamma_n)\frac{\log\gamma_n}{2\pi}=0.$$

One can also deduce that for a symmetric interval

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ -2\pi\beta/\log T\le\gamma'-\gamma\le 2\pi\beta/\log T}} 1 \sim \left(\int_{-\beta}^{\beta} 1-\left(\frac{\sin \pi x}{\pi x}\right)^2 dx + 1\right)\frac{T}{2\pi}\log T\,.$$

Letting $\beta \to 0$, we obtain

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma'=\gamma}} 1 \sim \frac{T}{2\pi}\log T\,.$$

But earlier we saw that

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma'=\gamma}} 1 = \sum_{0<\gamma\le T} m(\rho)\,.$$

It follows that

$$\sum_{\substack{0<\gamma\le T\\ \frac{1}{2}+i\gamma \text{ is simple}}} 1 \ge \sum_{0<\gamma\le T} (2-m(\rho))$$

$$= (2-1+o(1))\frac{T}{2\pi}\log T = (1+o(1))\frac{T}{2\pi}\log T\,.$$

In other words, almost all the zeros are simple.

D. Goldston and H. Montgomery [GM] have shown that the Pair Correlation Conjecture is equivalent to a certain estimate of the variance of the number of primes numbers in short intervals.

D. Goldston, S. G., and H. Montgomery have shown that it is also equivalent to an estimate for the mean–value

$$\int_0^T \left| \frac{\zeta'}{\zeta}(\sigma + it) \right|^2 \, dt \,,$$

for σ near $1/2$.

Estimates of $F(\alpha, T)$ when $\alpha \geq 1$ remain elusive. The only progress in this direction so far is the lower bound $F(\alpha, T) \geq 3/2 - \alpha + o(1)$ on the interval $(1, 3/2)$ under the assumption of the Generalized Riemannn Hypothesis. This is due to D. Goldston, S. G., A. Özlük, and C. Snyder.