The First 150 Years of the Riemann Zeta-Function

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I. Synopsis of Riemann’s paper

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse

( On the number of primes less than a given magnitude )
Figure: Riemann
Figure: First page of Riemann's paper
What Riemann proves

Riemann begins with Euler's observation that

\[\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (s > 1)\]

But he lets \(s = \sigma + it\) be complex. He denotes the common value by \(\zeta(s)\) and proves:

- \(\zeta(s)\) has an analytic continuation to \(\mathbb{C}\), except for a simple pole at \(s = 1\).
- The only zeros in \(\sigma < 0\) are simple zeros at \(s = -2, -4, -6, ...\).

\(\zeta(s)\) has a functional equation

\[\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-(1-s)/2} \Gamma((1-s)/2) \zeta(1-s)\]

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- \( \zeta(s) \) has a **functional equation**

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\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)
\]
What Riemann claims

\[ \zeta(s) \] has infinitely many nontrivial zeros \( \rho = \beta + i \gamma \) in the "critical strip" \( 0 \leq \sigma \leq 1 \).

If \( N(T) \) denotes the number of nontrivial zeros \( \rho = \beta + i \gamma \) with ordinates \( 0 < \gamma \leq T \), then as \( T \to \infty \),
\[
N(T) = T \frac{2}{\pi} \log T - T \frac{2}{\pi} + O(\log T).
\]

The function \( \xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s) \) is entire and has the product formula
\[
\xi(s) = \xi(0) \prod \rho \left( 1 - \frac{s}{\rho} \right).
\]

Here \( \rho \) runs over the nontrivial zeros of \( \zeta(s) \).
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What Riemann claims

Let \( \Lambda(n) = \log p \) if \( n = p^k \) and 0 otherwise. Then
\[
\psi(x) = \sum_{n \leq x} \Lambda(n) = x - \sum_{\rho} x/\rho + \infty \sum_{n=1}^\infty x/n^2 - \zeta'(0)/\zeta(0).
\]
(Riemann states this for \( \pi(x) = \sum p \leq x \) instead.)

Note that from this one can see why the Prime Number Theorem, \( \psi(x) \sim x \), might be true.
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The Riemann Hypothesis

Conjecture (The Riemann Hypothesis)

All the zeros $\rho = \beta + i\gamma$ in the critical strip lie on the line $\sigma = 1/2$. 

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II. Early developments after the paper
Hadamard 1893

Hadamard developed the theory of entire functions (Hadamard product formula) and proved the product formula for \( \xi(s) = \frac{1}{2} \frac{s}{s-1} \pi^{-s/2} \Gamma(s/2) \zeta(s) \).

\[ \xi(s) = \xi(0) \prod_{\rho} \left( 1 - \frac{s}{s-\rho} \right) = \xi(0) \prod_{\text{Im} \rho > 0} \left( 1 - \frac{s}{s-\rho} \right) \]

To do this, he proved the estimate \( N(T) \ll T \log T \), which is weaker than Riemann's assertion about \( N(T) \).
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which is weaker than Riemann’s assertion about \( N(T) \).
von Mangoldt proved Riemann's explicit formula for $\pi(x)$ and $\psi(x) = \sum_{n \leq x} \Lambda(n) = x - \sum_{\rho} \frac{x}{\rho} + \sum_{n=1}^{\infty} \frac{x}{n^2} - \frac{\zeta'(0)}{\zeta(0)}$. 
von Mangoldt proved Riemann’s explicit formula for $\pi(x)$ and

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Hadamard and de la Vallée Poussin independently proved the asymptotic form of the Prime Number Theorem, namely \( \psi(x) \sim x \). To do this, they both needed to prove that \( \zeta(1+it) \neq 0 \).
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de la Vallée Poussin 1899

De la Vallée Poussin proved the Prime Number Theorem with a remainder term:

\[ \psi(x) = x + O(xe^{-\sqrt{c_0 \log x}}). \]

This required him to prove that there is a zero-free region \( \sigma < 1 - c_0 \log t \).
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von Koch 1905 showed that the Riemann Hypothesis implies the Prime Number Theorem with a "small" remainder term:

\[ \psi(x) = x + O\left(\frac{x}{\log^2 x}\right) \]
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\[ \text{RH} \implies \psi(x) = x + O(x^{1/2} \log^2 x) \]
III. The order of $\zeta(s)$ in the critical strip
ζ(s) in the critical strip

The critical strip is the most important (and mysterious) region for ζ(s).

By the functional equation, it suffices to focus on $\frac{1}{2} \leq \sigma \leq 1$.

A natural question is: how large can ζ(s) be as t grows?

This is important because the growth of an analytic function and the distribution of its zeros are intimately connected.

The distribution of primes depends on it. Answers to other arithmetical questions depend on it.
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Implications of the size of $\zeta(s)$

Relation between growth and zeros: Jensen's Formula.

Let $f(z)$ be analytic for $|z| \leq R$ and $f(0) \neq 0$. If $z_1, z_2, \ldots, z_n$ are all the zeros of $f(z)$ inside $|z| \leq R$, then

$$\log(n|z_1 z_2 \cdots z_n|) = \frac{1}{2\pi} \int_{2\pi}^{0} \log|f(\text{Re}i\theta)| \, d\theta - \log|f(0)|.
$$

Example of an application to other problems: for $0 < c < 1$

$$\sum_{n \leq x} d_k(n) = xP_k - \frac{1}{2\pi} i \int_{c+i\infty}^{c-i\infty} \zeta_k(s)x^s \, ds.$$

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Implications of the size of $\zeta(s)$

Relation between growth and zeros:

$$\log\left(\frac{R^n|z_1 z_2 \cdots z_n|}{|f(0)|}\right) = \frac{1}{2\pi} \int_{0}^{2\pi} \log|f(\Re e^{i\theta})| \, d\theta.$$

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Example of an application to other problems: for $0 < c < 1$

$$\sum_{n \leq x} d_k(n) = x P_{k-1}(\log x) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta^k(s) \frac{x^s}{s} \, ds.$$
Estimates at the edge of the strip

Upper bounds for $\zeta(s)$ near $\sigma = 1$ allow one to widen the zero-free region. This leads to improvements in the remainder term for the PNT. For instance, we saw that de la Vallée Poussin showed that $\zeta(\sigma + it) \ll \log t$ in $\sigma \geq 1 - c_0 \log t$, and this implied that the $O$-term in the PNT is $\ll x e^{-\sqrt{c_1 \log x}}$.
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Estimates at the edge of the strip

\[ \zeta(\sigma + it) \ll \log t \log \log t \]

and no zeros in \( \sigma \geq 1 - c \log \log t \log t \)

\[ \Rightarrow O\text{-term in PNT} \ll x e^{-c \sqrt{\log x \log \log x}} \]

The idea is to approximate

\[ \zeta(\sigma + it) \approx N \sum_{n} \frac{1}{n^{\sigma + it}} \]

then use Weyl's method to estimate the exponential sums

\[ \sum_{a} e^{if(n)} \]
Estimates at the edge of the strip

Littlewood 1922

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\[ \sum_{a}^{b} n^{-it} = \sum_{a}^{b} e^{if(n)}. \]
Estimates at the edge of the strip

Vinogradov and Korobov 1958 (independently)

\[ \zeta(\sigma + it) \ll \log \frac{2}{3} t \]

and no zeros in \( \sigma \geq 1 - c \log \frac{2}{3} t \)

\[ \Rightarrow O \text{-term in PNT} \ll xe^{-c \log \frac{3}{5} - \epsilon x} \]

Where Littlewood used Weyl's method to estimate the exponential sums

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Vinogradov and Korobov 1958 (independently)

\[ \zeta(\sigma + it) \ll \log^{2/3} t \text{ and no zeros in } \sigma \geq 1 - \frac{c}{\log^{2/3} t} \]

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Where Littlewood used Weyl’s method to estimate the exponential sums

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Here is a summary:

\[ \zeta(1 + it) \ll \log t \quad (\text{de la Vallée Poussin}) \]

\[ \zeta(1 + it) \ll \log \log t \quad (\text{Littlewood-Weyl}) \]

\[ \zeta(1 + it) \ll \log \frac{2}{3} t \quad (\text{Vinogradov-Korobov}) \]

What should the truth be?

One can show that

\[ (1 + o(1)) e^{\gamma \log \log t} \leq i. \]

\[ |\zeta(1 + it)| \leq \text{RH} (1 + o(1)) e^{\gamma \log \log t}. \]
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What should the truth be? One can show that

\[
(1 + o(1)) e^{\gamma} \log \log t \leq_{i.o.} \left| \zeta(1 + it) \right| \leq_{RH} 2(1 + o(1)) e^{\gamma} \log \log t.
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Estimates inside the strip

Definition (Lindelöf 1908)

For a fixed $\sigma$, let $\mu(\sigma)$ denote the lower bound of the numbers $\mu$ such that $\zeta(\sigma + it) \ll (1 + |t|)\mu$. $\zeta(s)$ bounded for $\sigma > 1 \Rightarrow \mu(\sigma) = 0$ for $\sigma > 1$. 

$|\zeta(s)| \sim (|t|/2\pi)^{1/2 - \sigma} |\zeta(1 - s)| \Rightarrow \mu(\sigma) = 1/2 - \sigma + \mu(1 - \sigma)$.

In particular, $\mu(\sigma) = 1/2 - \sigma$ for $\sigma < 0$. 

(University of Rochester)
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(University of Rochester)
Lindelöf’s $\mu$-function

Lindelöf proved that $\mu(\sigma)$ is continuous, nonincreasing, and convex. These are in the same circle of ideas as the Phragmen-Lindelöf theorems. It follows that $\mu(1/2) \leq 1/4$, that is, $\zeta(1/2+\epsilon) \ll |t|^{1/4}+\epsilon$. This is a so-called convexity bound.
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This is a so called convexity bound.
Breaking convexity

Using Weyl's method of estimating exponential sums, Hardy and Littlewood showed that
\[ \zeta(1/2 + it) \ll |t|^{1/6 + \epsilon}. \]

The best results for \( \mu(\sigma) \) since have come from exponential sum methods: van der Corput, Vinogradov, Kolesnik, Bombieri-Iwaniec, Huxley-Watt.

Huxley and Watt show that \( \mu(\sigma) < 9/56. \)

Conjecture (Lindelöf)
\[ \mu(\sigma) = 0 \text{ for } \sigma \geq 1/2. \]
That is, \( \zeta(1/2 + it) \ll |t|^{\epsilon} \) for \( t \) large.
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What we expect the order to be

The LH says that for large $|t| \log |ζ(1/2+it)| \leq ϵ \log |t|$.

It is also known that $√c \log t \log \log t \leq i \circ \log |ζ(1/2+it)| \ll RH \log t \log \log t$.

Which bound, the upper or the lower, is closest to the truth is one of the important open questions.
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IV. Mean value theorems
Mean value theorems

Averages such as \[ \int_0^T |\zeta(\sigma + it)|^2 \, dt \] have been another main focus of research because averages as well as pointwise upper bounds tell us about zeros and have other applications. Mean values are easier to prove than pointwise bounds. The techniques developed to treat them have proved important in other contexts.
Averages such as $\int_0^T |\zeta(\sigma + it)|^{2k} dt$ have been another main focus of research.

Mean value theorems

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Mean value theorems
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Landau 1908

\[ \int_{0}^{T} |\zeta(\sigma + it)|^2 dt \sim \zeta(2\sigma) T \quad (\sigma > 1/2 \text{ fixed}). \]
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**Hardy-Littlewood 1918**

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\int_0^T |\zeta(1/2 + it)|^2 \, dt \sim T \log T.
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Mean value theorems

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For this H-L developed the approximate functional equation

\[
\zeta(s) = \sum_{n \leq \sqrt{t/2\pi}} n^{-s} + \chi(s) \sum_{n \leq \sqrt{t/2\pi}} n^{s-1} + O(\ldots),
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Mean value theorems

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which has proved an extremely important tool ever since.
Mean value theorems

Hardy-Littlewood 1918

\[ \int_{0}^{T} |\zeta(\sigma + it)|^4 \, dt \sim \zeta(4\sigma) T (\sigma > 1/2 \text{ fixed}). \]

Ingham 1926

\[ \int_{0}^{T} |\zeta(1/2 + it)|^4 \, dt \sim \frac{T^2}{2\pi} \log^4 T. \]

This was done by using an approximate functional equation for \( \zeta^2(s) \).

When \( k \) is a positive integer, Ramachandra showed that

\[ \int_{0}^{T} |\zeta(1/2 + it)|^{2k} \, dt \gg T \log^2 k T. \]

This is believed to be the correct upper bound as well.
Mean value theorems

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Mean value theorems

This suggests the problem of determining constants $C_k$ such that
\[ \int_0^T |\zeta(1/2 + it)|^2 k \, dt \sim C_k T \log^2 k. \]

Conrey-Ghosh suggested that $C_k = a_k g_k \Gamma(k/2 + 1)$, where $a_k = \prod_p \left(1 - \frac{1}{p} \right)^{k/2} \sum_{r=0}^{\infty} d_2(k)(p^r) p^{r/2}$ and $g_k$ is an integer.
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\[ \int_0^T |\zeta(1/2 + it)|^2 k \, dt \sim a_k g_k \Gamma(k^2 + 1) T \log k^2 T, \]

\[ g_1 = 1 \text{ and } g_2 = 2 \] are known.

Conrey and Ghosh conjectured that \( g_3 = 42 \).

Conrey and G conjecured that \( g_4 = 24024 \).

Keating and Snaith used random matrix theory to conjecture the value of \( g_k \) for every value of \( k > -1/2 \).

Soundararajan has recently shown that on RH

\[ \int_0^T |\zeta(1/2 + it)|^2 k \, dt \ll T \log k^2 + \epsilon T. \]
Mean value theorems

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V. Zero-density estimates
Zero-density estimates

Let $N(\sigma, T)$ denote the number of zeros of $\zeta(s)$ with abscissae to the right of $\sigma$ and ordinates between 0 and $T$.

Zero-density estimates are bounds for $N(\sigma, T)$ when $\sigma > 1/2$.

Bohr and Landau 1912 showed that for each fixed $\sigma > 1/2$, $N(\sigma, T) \ll T$.

Since $N(T) \sim (T/2\pi) \log T$, this says the proportion of zeros to the right of $\sigma > 1/2$ tends to 0 as $T \to \infty$. 

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$$N(T) \sim \left( \frac{T}{2\pi} \right) \log T,$$
Zero-density estimates

Let \( N(\sigma, T) \) denote the number of zeros of \( \zeta(s) \) with abscissae to the right of \( \sigma \) and ordinates between 0 and \( T \).

Zero-density estimates are bounds for \( N(\sigma, T) \) when \( \sigma > 1/2 \).

Bohr and Landau 1912 showed that for each fixed \( \sigma > 1/2 \),

\[
N(\sigma, T) \ll T.
\]

Since

\[
N(T) \sim (T/2\pi) \log T,
\]

this says the proportion of zeros to the right of \( \sigma > 1/2 \) tends to 0 as \( T \to \infty \).
Zero-density estimates

Bohr and Landau used Jensen's formula and
\[\int_0^T |\zeta(\sigma + it)|^2 dt \ll T \quad (\sigma > 1/2 \text{ fixed})\]
to prove this.

Today we have much better zero-density estimates of the form
\[N(\sigma, T) \ll T^{\theta(\sigma)}\]
with \(\theta(\sigma)\) strictly less than 1.

The conjecture that
\[N(\sigma, T) \ll T^{2\left(1 - \sigma\right) + \epsilon}\]
is called the Density Hypothesis.

Obviously RH implies the Density Hypothesis.

LH implies
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VI. The distribution of $a$-values of $\zeta(s)$
The distribution of \( a \)-values of \( \zeta(s) \)

What can we say about the distribution of non-zero values, \( a \), of the zeta-function?

A lovely theory due mostly to H. Bohr developed around this question. Here are two results.

First, the curve

\[
f(t) = \zeta(\sigma + it) \quad \left( \frac{1}{2} < \sigma \leq 1 \text{ fixed}, \; t \in \mathbb{R} \right)
\]

is dense in \( \mathbb{C} \).

The idea is to show that

\[
\zeta(\sigma + it) \approx \prod_{p \leq N} \left( 1 - p^{-\sigma} - it \right)^{-1}
\]

for most \( t \).

Use Kronecker's theorem to find a \( t \) so that the numbers \( p - it \) point in such a way that

\[
\prod_{p \leq N} \left( 1 - p^{-\sigma} - it \right)^{-1} \approx a.
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\hspace{1cm} use Kronecker's theorem to find a $t$ so that the numbers $p - it$ point in such a way that\
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(University of Rochester)
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As a second result, let $N_a(\sigma_1, \sigma_2, T)$ be the number of solutions of $\zeta(s) = a$ in the rectangular area $\sigma_1 \leq \sigma \leq \sigma_2, 0 \leq t \leq T$.

Suppose that $1/2 < \sigma_1 < \sigma_2 \leq 1$. Then there exists a positive constant $c(\sigma_1, \sigma_2)$ such that $N_a(\sigma_1, \sigma_2, T) \sim c(\sigma_1, \sigma_2)T$.

Notice that this is quite different from the case $a = 0$, because modern zero-density estimates imply $N_0(\sigma_1, \sigma_2, T) \ll T^{1-\theta}$ ($\theta < 1$).
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VII. Number of zeros on the line as $T \to \infty$
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Let $N_0(T) = \# \{1/2 + i\gamma : \xi(1/2 + i\gamma) = 0, 0 < \gamma < T\}$ denote the number of zeros on the critical line up to height $T$.

The important estimates were

- Hardy 1914: $N_0(T) \to \infty$ (as $T \to \infty$)
- Hardy-Littlewood 1921: $N_0(T) > c T$
- Selberg 1942: $N_0(T) > c N(T)$
- Levinson 1974: $N_0(T) > \frac{1}{3} N(T)$
- Conrey 1989: $N_0(T) > \frac{2}{5} N(T)$

These all rely heavily on mean value estimates.
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These all rely heavily on mean value estimates.
One can write the functional equation as
\[ \zeta(s) = \chi(s) \zeta(1-s), \]
or as
\[ \chi - 1/2(s) \zeta(s) = \chi 1/2(s) \zeta(1-s). \]
Then
\[ Z(t) = \chi - 1/2(1/2 + it) \zeta(1/2 + it) \]
has the same zeros as \( \zeta(s) \) on \( \sigma = 1/2 \) and is real.
If \( Z(t) \) had no zeros for \( t \geq T_0 \), the integrals
\[ \left| \int_{T}^{T_0} Z(t) \, dt \right| \quad \text{and} \quad \int_{T}^{T_0} |Z(t)| \, dt \]
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VIII. Calculations of zeros on the line
Numerical calculations of zeros

Gram 1903
The zeros up to 50 (the first 15) are on the line and simple.

Backlund 1912
The zeros up to 200 are on the line.

Hutchison 1925
The zeros up to 300 are on the line.

Titchmarsh, Turing, Lehman, Brent, van de Lune, te Riele, Odlyzko, Wedeniwski, ...

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The first $10^{13}$ (ten trillion) zeros are on the line.

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**Gourdon-Demichel 2004** The first $10^{13}$ (ten trillion) zeros are on the line. Moreover, billions of zeros near the $10^{24}$ zero are on the line.
IX. More recent developments
A major theme of research over the last 35 years has been to understand the distribution of the zeros on the critical line assuming that the Riemann Hypothesis is true. In 1974 Montgomery conjectured that the zeros are distributed like the eigenvalues of random Hermitian matrices. From 1980 on Odlyzko did a vast amount of numerical calculation that strongly supported Montgomery's conjecture.
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New mean value theorems

G and Conrey, Ghosh, and G proved a number of discrete mean value theorems of the type

\[ \sum_{0 < \gamma \leq T} |\zeta(\rho + i\alpha)|^2 \]

and

\[ \sum_{0 < \gamma \leq T} |\zeta'(\rho)| M_N(\rho)|^2, \]

where \( \rho = 1/2 + i\gamma \) runs over the zeros.

Assuming RH and sometimes GLH and GRH, Conrey, Ghosh, and G used these to prove that

there are large and small gaps between consecutive zeros.

over 70% of the zeros are simple.
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Assuming RH and sometimes GLH and GRH, Conrey, Ghosh, and G used these to prove that

- there are large and small gaps between consecutive zeros.
- over 70% of the zeros are simple.
A major development was Keating and Snaith's modeling of \( \zeta(s) \) by the characteristic polynomials of random Hermitian matrices. It allowed them to determine the constants \( g_k \) in
\[
\int_0^T |\zeta(1/2 + it)|^2 k \, dt \sim a_k g_k \Gamma(k^2 + 1) T \log k^2 T.
\]
It has had applications to elliptic curves, for example. Hughes, Keating, O'Connell used it to conjecture the discrete means
\[
\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^2 k
\]
Mezzadri used it to study the distribution of the zeros of \( \zeta'(s) \).
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Lower order terms and ratios

The Keating-Snaith results led to the quest for the lower order terms in the asymptotic expansion of the moments. This resulted in the discovery of new heuristics for the moments not involving RMT. It also led to heuristics for very general moment questions (the so-called "ratios conjecture").

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A hybrid formula

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G. Hughes, Keating found an unconditional hybrid formula for $\zeta(s)$. It says (roughly) that

$$\zeta(s) = \prod_{p \leq X} \left(1 - \frac{1}{p} - \frac{s}{p} \right)^{-1} \prod_{|s - \rho| \leq 1/\log X} \left(1 - X^{s - \rho} e^{\gamma} \right)$$

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It also explains why the constant in the moment splits as $\frac{a_k g_k}{\Gamma(k^2 + 1)}$. 
Finally, the hybrid formula has led to conjectural answers to the deep question of the exact order of $\zeta(s)$ in the critical strip. Recall that $\left(1 + o(1)\right) e^{\gamma \log \log t} \leq i \cdot |\zeta(1 + it)| \leq \text{RH} 2 \left(1 + o(1)\right) e^{\gamma \log \log t}$, so that a factor of 2 is in question. Arguments from the hybrid model suggest that the 2 should be dropped.
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