

Zeta functions of Artin Stacks over \mathbb{F}_p

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1. Recall the theory of zeta functions of algebraic varieties over a finite field

- $X_0 = \text{alg. variety } / \mathbb{F}_p$, given by some polynomial equations $f_j(t_1, \dots, t_n) = 0$ over \mathbb{F}_p .
- $X_0(\mathbb{F}_{p^\nu}) = \text{set of solutions of } f_j(t_i) = 0 \text{ over } \mathbb{F}_{p^\nu}$.
This is a finite set.
- Def: $Z(X_0, t) := \exp\left(\sum_{v=1}^{\infty} \#X_0(\mathbb{F}_{p^\nu}) \cdot \frac{t^\nu}{\nu}\right) \in \mathbb{Q}[[t]]$.
- Thm. (Dwork, Grothendieck, Deligne, ...) Let X_0 / \mathbb{F}_p be an alg. variety of dimension d . Then
 - (a) $Z(X_0, t)$ is a RATIONAL function in t , i.e., $\in \mathbb{Q}(t)$.
 - (b) Let $H^i_c(X, \mathbb{Q}_\ell)$ be the ℓ -adic cohomology with compact support. It has a \mathbb{Q}_ℓ -linear operator F acting on it, the geometric Frobenius $\in \text{Gal}(\overline{\mathbb{F}}_p / \mathbb{F}_p) = \widehat{\mathbb{Z}}$. Let $P_{i,\ell}(X_0, t) = \det(1 - Ft, H^i_c(X, \mathbb{Q}_\ell)) \in \mathbb{Q}_\ell[[t]]$.

Then

$$Z(X_0, t) = \frac{P_{1,\ell}(t) \cdot P_{3,\ell}(t) \cdots P_{2d-1,\ell}(t)}{P_{0,\ell}(t) \cdot P_{2,\ell}(t) \cdots P_{2d,\ell}(t)}$$

(c). Fix any isomorphism of fields $\varphi: \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, and factor ${}_2P_{i,\ell}(t)$ in \mathbb{C} :

$${}_2P_{i,\ell}(t) = \prod_j (1 - \alpha_{ij} t), \quad \alpha_{ij} \in \mathbb{C}.$$

Then the α_{ij} 's are algebraic integers, and $|\alpha_{ij}|_C = p^{w/2}$
 for some integer $w \in \mathbb{Z}$, and $0 \leq w \leq i$. \square

Known as the Weil Conjecture.

- Cor. (Deligne) If in addition X_0 is proper (something like compact) and smooth (like a manifold $/\mathbb{R}, \mathbb{C}$), then the coefficients of $P_{i,l}(X_0, t)$ are rational integers (i.e., $\in \mathbb{Z}$) and are independent of l , for any prime number $l \neq p$. \square

2. Examples.

- $X_0 = \mathbb{P}^1$, the projective line. $\#X_0(\mathbb{F}_{p^v}) = p^v + 1$, so

$$Z(\mathbb{P}^1, t) = \exp \left(\sum_{v=1}^{\infty} \frac{(pt)^v}{v} + \sum_{v=1}^{\infty} \frac{t^v}{v} \right) = \frac{1}{(1-t)(1-pt)}.$$

Here $P_{0,l}(t) = 1-t$, $P_{1,l}(t) = 1$, $P_{2,l}(t) = 1-pt$. —
 they are all indep. of l .

- $X_0 = E$, an elliptic curve $/\mathbb{F}_p$.

Thm. (Hasse, 1930s) Let α and β be roots of the equation

$$T^2 - aT + p = 0, \quad a = p+1 - \#E(\mathbb{F}_p) \in \mathbb{Z}$$

Then $\#E(\mathbb{F}_{p^v}) = p^v + 1 - \alpha^v - \beta^v$ for any $v \geq 1$, and

$$|\alpha| \leq 2\sqrt{p} \quad (\text{equiv, } |\alpha| = |\beta| = \sqrt{p}). \quad \square$$

$$\text{So } Z(E, t) = \frac{1 - at + pt^2}{(1-t)(1-pt)} = \frac{P_{1,l}(t)}{P_{0,l}(t) \cdot P_{2,l}(t)}$$

3. Generalization to Artin stacks $/\mathbb{F}_p$.

- Stacks is a vast generalization of algebraic varieties. Basically they look like quotients U/G , where U is a variety and

G is an algebraic group acting on \mathcal{U} .

- Idea: If a point on a stack has an automorphism group of order 2 acting on it, then the point should be considered as $\frac{1}{2}$ point! For a stack X_0/\mathbb{F}_p , use this to define $\#X_0(\mathbb{F}_{p^v})$, which is a rational number in general.

Define

$$Z(X_0, t) = \exp\left(\sum_{v=1}^{\infty} \#X_0(\mathbb{F}_{p^v}) \frac{t^v}{v}\right) \in \mathbb{Q}[[t]].$$

(multiplicative group, like \mathbb{C}^\times)

4. Examples.

- $X_0 = BG_m$, the classifying stack of $G_m = [P^1/G_m]$.

$$\#BG_m(\mathbb{F}_{p^v}) = \frac{1}{\#G_m(\mathbb{F}_{p^v})} = \frac{1}{p^v - 1} = \frac{p^{-v}}{1 - p^{-v}} =$$

$$p^{-v} + p^{-2v} + p^{-3v} + p^{-4v} + \dots, \text{ so}$$

$$Z(BG_m/\mathbb{F}_p, t) = \frac{1}{(1 - \frac{t}{p})(1 - \frac{t}{p^2})(1 - \frac{t}{p^3})(1 - \frac{t}{p^4}) \dots}$$

- Q1: Is the infinite product converges term by term?

- Q2: The power series $Z(BG_m, t) = \exp\left(\sum_{v \geq 1} \frac{t^v}{(p^v - 1)v}\right)$ converges in the open disk $|t| < p$. Does it have meromorphic continuation to the whole complex t -plane?

Observations:

- $P_{-2-2n, l}(BG_m, t) = 1 - \frac{t}{p^{n+1}}$ for all $n \geq 0$, and

other $P_{l, l}(BG_m, t) = 1$. They are indep. of l , and the "dij" for $P_{-2-2n}(t)$ is $\frac{1}{p^{n+1}}$, $|\frac{1}{p^{n+1}}| = p^{w/2}$, $w = -2-2n \leq -2-2n$.

- BE , the classifying stack of an elliptic curve / \mathbb{F}_p .

$$\#\text{BE}(\mathbb{F}_{p^\nu}) = \frac{1}{\#E(\mathbb{F}_{p^\nu})} = \frac{1}{p^\nu + 1 - \alpha^\nu - \beta^\nu} = \frac{1}{(1-\alpha^\nu)(1-\beta^\nu)}.$$

Can show the answers to Q1 and Q2 to BG_m and BE are both YES. Moreover, BE is also "indep. of l ".

BUT: the F -eigenvalues on $H_c^{-2-2n}(\text{BE}, \mathbb{Q}_\ell)$, $n \geq 0$, are

$p^{-1}\alpha^{-n}, p^{-1}\alpha^{-n}\beta^{-1}, \dots, p^{-1}\beta^{-n}$, and their absolute values are $p^{-1} \cdot p^{-\frac{n}{2}} = p^{\frac{w}{2}}$, where $w = -n-2$, which is NOT $\leq -2-2n$ unless $n=0$.

↑ the index of the
cohom. H_c^{-2-2n}

• Q4: Do we always have "indep. of l " for stacks?

• Q3: Any upper bound for w that is valid for any stacks?

5. Thm: For any Artin stack \mathcal{X}_0 (all stacks that we are interested in for arithmetic geometry) satisfying some finite type condition (also usually satisfied), we have

(1). Q1 holds: $\prod_{i \in \mathbb{Z}} P_{i,\ell}(\mathcal{X}_0, t)^{(-1)^{i-1}}$ converges term by term.

(2). Q2 holds: $\sum (\mathcal{X}_0, t) = \prod_i P_{i,\ell}(t)^{(-1)^{i-1}}$ as power series

in $\mathbb{Q}_\ell[[t]]$, and for any embedding $\tau: \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$, the infinite product expression gives a meromorphic continuation of the function on the left to the whole complex plane.

(3). A \mathbb{Z} -mixed sheaf \mathcal{F}_0 on \mathcal{X}_0 (e.g. $\mathcal{F}_0 = \mathbb{Q}_\ell$) of \mathbb{Z} -weights $\leq w \in \mathbb{R}$ (for \mathbb{Q}_ℓ the \mathbb{Z} -wt. is 0), the compact cohomology

$H_c^i(X, \mathbb{Q}_\ell)$ is 2-mixed of wts $\leq \frac{i}{2} + \dim(X_0) + n$.

(4). Let X_0 be a proper smooth variety/ \mathbb{F}_p , G_0 a linear alg. group acting on X_0 , the quotient stack $[X_0/G_0]$ satisfies "indep. of ℓ ". \square .

E.g., for $X_0 = BE$ of dim -1 , $i = -2-2n$, the 2-weight of $H_c^{-2-2n}(BE, \mathbb{Q}_\ell)$ is $-2-n$, which is $(\leq) \frac{-2-2n}{2} + (-1)$.

$$\frac{i}{2} + \dim BE.$$

6. Why do we care about this?

- N. Katz^(?) conjectured "indep. of ℓ " for alg. varieties/ \mathbb{F}_p , which provides evidence for "motives". We hope the same should be true for stacks.
- Given an alg. variety X over \mathbb{Q} (\mathbb{C} or over \mathbb{Z}), we have the Hasse - Weil zeta function of X :

$$\zeta(X, s) = \prod_p Z(X(\text{mod } p)/\mathbb{F}_p, t = p^{-s})$$

$$= \prod_{i=0}^{2d} \left[\prod_{j=1}^{(f-1)^{i+1}} \text{diag}(t^j) \right] \underbrace{\prod_{j=1}^{(f-1)^{i+1}}}_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} L(H_c^i(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell), s)^{(-1)^i},$$

where

$$L(H_c^i(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell), s) = \prod_p \det(1 - \text{Frob}_p \cdot p^{-s}, H_c^i(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_P})^{-1}.$$

Note that unlike complex representations of Galois group, these ℓ -adic representations of Galois group almost never factor through a finite quotient, hence we can't use Brower's thm.

Hasse-Weil's conj: $\mathcal{F}(X, s)$ has analytic continuation and (meromorphic) functional equation of expected form. Also Riemann hypothesis.

Conj. (Langlands) $\mathcal{F}(X, s)$ is "modular", i.e., an alternating product of automorphic L-functions.

Can conjecture the same for stacks "since"

$$\{\text{Stacks}\} \subset \left\{ \begin{array}{l} \text{motives,} \\ \text{spec. seg.} \\ \text{resol. of sing.} \end{array} \right\} \xrightarrow{\text{motivic}} \text{Langlands} \subset \left\{ \begin{array}{l} \text{automorphic} \\ \text{L-fns} \end{array} \right\}$$

Wiles, Taylor (1994): Elliptic curves E/\mathbb{Q} are modular.
et. al. (2001)

- Hope: the theory of stacks can help proving things about varieties. For instance, some singular varieties are smooth as stacks.

- Let $f(t) \in \mathbb{Q}[t]$ be irreducible, $k = \mathbb{Q}[t]/(f(t))$, a number field. Also the polynomial $f(t)$ defines an alg. variety $/\mathbb{Q}$ (i.e., $X = \text{Spec}(k)$). Then

$$\mathcal{F}(\text{Spec}(k), s) = \zeta_k(s), \text{ the Dedekind zeta fcn.}$$

So the theory of arithmetic geometry (studying alg. var. or stacks over \mathbb{F}_p or \mathbb{Q}) can be regarded as a vast generalization of classical number theory (studying number fields).