The Ideal Sieve

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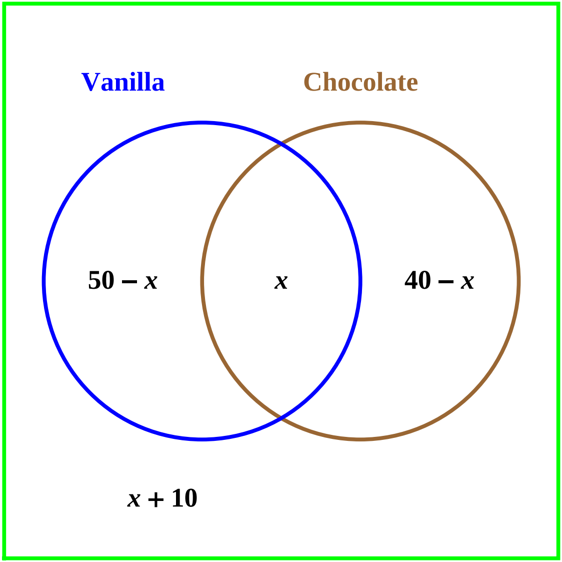
World's Easiest Sieve Problem

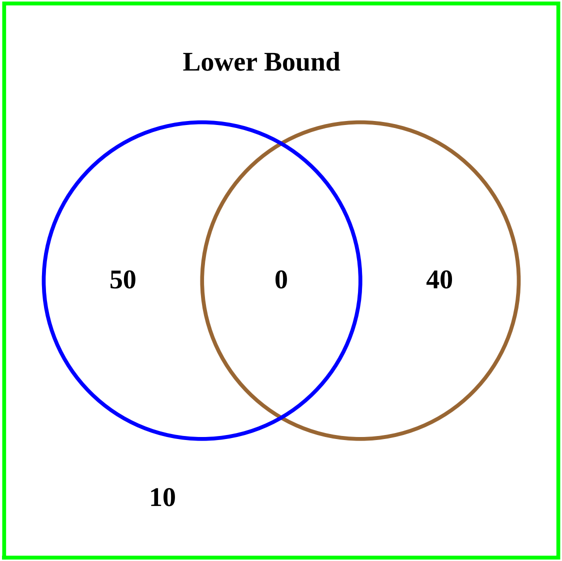
A survey of 100 people finds that

1. 50 people like vanilla ice cream and
2. 40 people like chocolate ice cream.

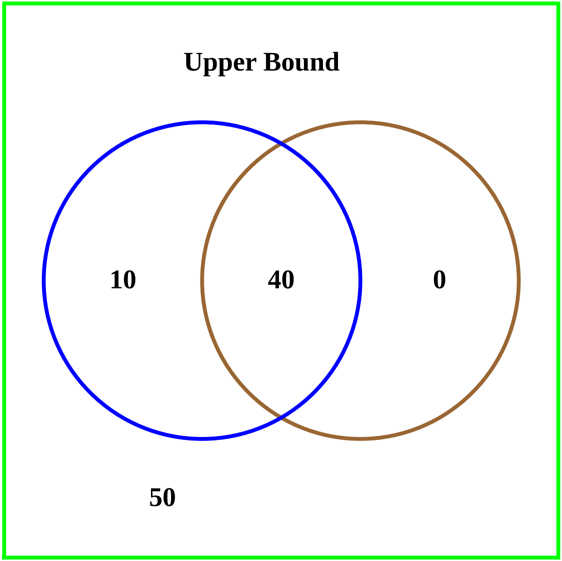
How many people don't like either flavor?



$0 \leq x \leq 40.$



$x = 0$: At least 10 people don't like either chocolate or vanilla.



$x = 40$: At most 50 people don't like either chocolate or vanilla.

Let $\mathcal{A}$ be a finite set of integers, and assume each $n \in \mathcal{A}$ is equipped with a non-negative weight w_n.

Let $\mathcal{P}$ be a finite set of primes and

$$P = \prod_{p \in \mathcal{P}} p.$$

The sieve problem: Get upper and lower bounds for

$$S_1 = \sum_{\substack{n \in \mathcal{A} \\ (n,P)=1}} w_n,$$

from information about

$$W_d = \sum_{\substack{n \in \mathcal{A} \\ d|n}} w_n.$$

Estimates for W_d take the form

$$g(d)X - R_d^- \le W_d \le g(d)X + R_d^+$$

for $d|P$, where g is a non-negative multiplicative function.

λ_d an *upper bound sieve* if for every $n \in \mathcal{A}$,

$$\sum_{d|(n,P)} \lambda_d \geq \begin{cases} 1 & \text{if } (n,P)=1, \\ 0 & \text{otherwise.} \end{cases}$$

For an upper bound sieve,

$$S_1 \leq \sum_{n \in \mathcal{A}} w_n \sum_{d|(n,P)} \lambda_d \leq X \sum_{d|P} g(d) + \sum_{d|P} |\lambda_d| R_d^{\operatorname{sgn} \lambda_d}.$$

λ_d an *lower bound sieve* if for every $n|P$,

$$\sum_{d|n} \lambda_d \leq \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For a lower bound sieve,

$$S_1 \leq \sum_{n \in \mathcal{A}} w_n \sum_{d|(n,P)} \lambda_d \geq X \sum_{d|P} \lambda_d g(d) - \sum_{d|P} |\lambda_d| R_d^{-\operatorname{sgn} \lambda_d}.$$

The Ideal Sieve

In practice, the error terms are controlled by requiring $\lambda_d = 0$ for $d \geq z$, where z is some appropriate parameter.

We idealize this situation: Assume

$$R_d = \begin{cases} 0 & \text{if } d < z \\ \infty & \text{if } d \geq z \end{cases}$$

Therefore, the only useful sieves have $\lambda_d = 0$ for $d \geq z$.

If $d < z$, then

$$W_d = \sum_{\substack{n \in \mathcal{A} \\ d|n}} w_n = g(d)X$$

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By homogeniety, we may normalize to $X = 1$.

Looking for Extremal Examples

The key idea: Make a change of basis. Recall

$$W_d = \sum_{\substack{n\in\mathcal{A} \\ d|n}} w_n = g(d).$$

Define

$$S_d = \sum_{\substack{n\in\mathcal{A} \\ (n,P)=d}} w_n.$$

Then

$$W_d = \sum_{e|\frac{P}{d}} S_{de}, \quad S_d = \sum_{e|\frac{P}{d}} \mu(e) W_{de}.$$

Now describe sets in terms of S_d instead of W_d or w_n.

Admissible Sets

The set $\{S_d : d|P\}$ is admissible if

1. $S_d \geq 0$ for all $d|P$, and
2. if $d|P$ and $d \leq z$, then $\displaystyle\sum_{\delta|\frac{P}{d}} S_{\delta d} = g(d)$.

If $\{\lambda_d\}$ is an upper bound sieve, then for any $\mathcal{A}$,

$$S_1 \leq \sum_{n\in\mathcal{A}} w_n \sum_{d|n} \lambda_d = \sum_{d|P} \lambda_d g(d).$$

If we can find an admissible set $\{S_d\}$ such that

$$S_1 = \sum_{d|P} \lambda_d g(d),$$

then $\{\lambda_d\}$ is optimal.

Linear Programming

The situation here is one of linear programming.
By the fundamental duality theorem

$$\max_{w_n} S_1 = \min_{\lambda_d \in \mathcal{L}^+} \sum_{d|P} g(d)\lambda_d$$

where $\mathcal{L}^+$ denotes the set of all upper bound sifting functions.
Similarly,

$$\min_{w_n} S_1 = \max_{\lambda_d \in \mathcal{L}^-} \sum_{d|P} g(d)\lambda_d$$

where $\mathcal{L}^-$ denotes the set of all lower bound sifting functions.

Define $\theta_m = \sum_{d|m} \lambda_d$. The condition for an upper bound sieve may be rephrased as

$$\theta_1 \geq 1, \quad \theta_m \geq 0 \text{ for } m > 1.$$

By Möbius inversion,

$$\lambda_d = \sum_{m|d} \mu\left(\frac{d}{m}\right) \theta_m$$

so knowing θ is equivalent to knowing λ.

Our Basic Problem

Assume that the sifting primes p lie in the range

$$z^{\alpha} < p \le z^{\beta}.$$

Identify best possible upper and lower bound sieves, and identify extremal examples.

Sifting primes in $(z^{1/2}, z]$

The sieving primes are $p_1, \ldots, p_K$ with

$$z^{1/2} < p_i \leq z.$$

The product of any two of these primes exceeds z, so $\lambda_d = 0$ if d has two or more prime factors.

Lower Bound:

Let

$$\lambda_1 = 1, \quad \lambda_p = -1.$$

If $n|P$, then

$$\theta_n = \sum_{d|n} \lambda_d = 1 - \omega(n),$$

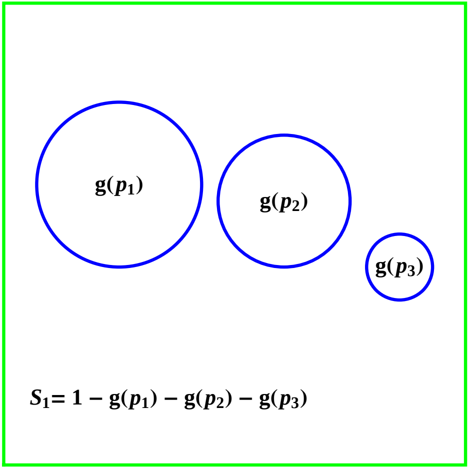
so we have a lower bound sieve.
Therefore

$$S_1 \geq 1 - \sum_{p|P} g(p)$$

Optimality of Lower Bound

Take

$$S_1 = 1 - \sum_{p|P} g(p), \quad S_p = g(p).$$



Upper Bound:

Let p_1 be a prime such that

$$g(p_1) \geq g(p_i)$$

for all i. Take

$$\lambda_1 = 1, \quad \lambda_{p_1} = -1,$$

and $\lambda_d = 0$ otherwise.
If $n|P$, then

$$\theta_n = \sum_{d|n} \lambda_d = \begin{cases} 0 & \text{if } p_1|n, \\ 1 & \text{if } p_1 \nmid n. \end{cases}$$

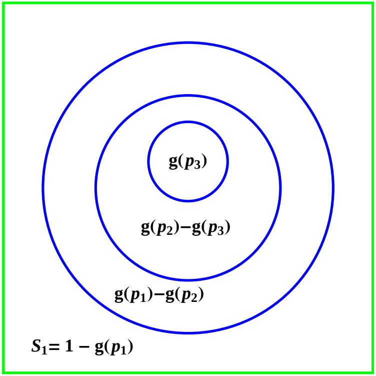
so we have an upper bound sieve.
Therefore

$$S_1 \leq 1 - g(p_1).$$

Optimality of Upper Bound

Arrange the primes so that $g(p_1) \geq g(p_2) \ldots \geq g(p_K)$.

$$S_1 = 1 - g(p_1), S_{p_1} = g(p_1) - g(p_2),$$
$$S_{p_1 p_2} = g(p_2) - g(p_3), \ldots, S_{p_1 p_2 \ldots p_K} = g(p_K).$$



Sifing Primes in $[z^{1/3}, z^{1/2})$

Suppose that the sifting primes $\mathcal{P} \subseteq [z^{1/3}, z^{1/2})$.
In other words, for all primes in $\mathcal{P}$,

$$z^{1/3} \leq p < z^{1/2}.$$

We must have $\lambda_d = 0$ if $\omega(d) \geq 3$.
Suppose also that (dimension assumption)

$$\sum_{p \in \mathcal{P}} g(p) = \kappa + o(1), \text{ and } \sum_{p \in \mathcal{P}} g^2(p) = o(1)$$

as $z \to \infty$.
Here are three approaches to finding an upper bound for S_1.

A combinatorial sieve has $\lambda_d = \pm 1$ or 0.
Define $\lambda_p = -1$ if $z^{1/3} \leq p < y$ and $\lambda_p = 0$ if $p > y$.
Define $\lambda_{pq} = \lambda_p \lambda_q$. Then

$$\theta_n = \sum_{d|n} \lambda_d = (1 - \ell)\left(1 - \frac{\ell}{2}\right) \geq 0,$$

where ℓ is the number of prime factors of n not exceeding y.
Thus

$$S_1 \lesssim 1 - t + t^2/2$$

for some t, $0 \leq t \leq \kappa$.

If $0 < \kappa \leq 1$, take $t = \kappa$: $S_1 \lesssim 1 - \kappa + \frac{1}{2}\kappa^2$.

If $1 < \kappa$, take $t = 1$: $S_1 \lesssim 1/2$.

Λ^2 sieve

$$S_1 \leq \sum_{n \in \mathcal{A}} w_n \left(\sum_{d|n} \lambda_d \right)^2 = \sum_{d,e} \lambda_d \lambda_e g([d,e])$$

Say $\lambda_p = a$ if $p \in \mathcal{P}$.

Then

$$S_1 \lesssim (1 + \kappa a)^2 + \kappa a^2.$$

Optimal choice is $a = -1/(\kappa + 1)$, and this gives

$$S_1 \lesssim \frac{1}{\kappa + 1}.$$

This is better than the combinatorial sieve iff $\kappa > 1$.

Optimal Sieve

Consider those λ where λ_d depends only on $\omega(d)$.

Write $\lambda_d = \lambda(\ell)$ when $\omega(d) = \ell$.

Write $\theta_d = \sum_{e|d} \lambda_e = \theta(\ell)$ when $\omega(d) = \ell$.

Need $\theta(0) \geq 1, \theta(\ell) \geq 0$. Take

$$\theta(\ell) = \left(1 - \frac{\ell}{r}\right)\left(1 - \frac{\ell}{r+1}\right).$$

where r is a positive integer chosen later.

Then

$$S_1 \lesssim 1 - \frac{2\kappa}{r+1} + \frac{\kappa^2}{r(r+1)} = \frac{(r-\kappa)^2 + r}{r(r+1)}.$$

The optimal choice is $r = [1+\kappa]$.
When κ is an integer,

$$S_1 \lesssim \frac{1}{\kappa+1},$$

which is the same as Λ^2.

When $0 < \kappa \leq 1$,

$$S_1 \lesssim 1 - \kappa + \frac{1}{2}\kappa^2,$$

which is the same as the combinatorial sieve.

To show this upper bound is optimal, take

$$S_d = \begin{cases} \frac{(r-1)!}{\kappa^{r-1}}(r-\kappa)g(d) & \text{if } \omega(d) = r, \\ \frac{r!}{\kappa^r}(\kappa - r + 1)g(d) & \text{if } \omega(d) = r+1, \\ \frac{(r-\kappa)^2+r}{r(r+1)} & \text{if } d = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lower Bound for $R = 3$

When the sifting primes p satisfy $z^{1/4} \le p < z^{1/3}$, the optimal lower bound θ is

$$\theta(\ell) = (1-\ell)\left(1 - \frac{\ell}{r}\right)\left(1 - \frac{\ell}{r+1}\right),$$

and the corresponding lower bound is

$$S_1 \gtrsim 1 - \kappa + \frac{(2r-1)\kappa^2}{(r+1)r} - \frac{\kappa^3}{(r+1)r}$$

with $r = [\kappa + 2]$. The right-hand side is positive for $\kappa < 2$.

In "Lectures on Sieves" (Collected Works II), Selberg considers a more general problem where the primes in $\mathcal{P}$ satisfy

$$z^{1/(R+1)} \leq p < z^{1/R}$$

for arbitrary integer $R \geq 1$.

He proves that in an optimal sieve, λ_d depends only on $\omega(d)$, but the proof does not identify extremal examples.

Let

$$v_R = \sup\{\kappa : S_1 > 0\}.$$

Then $v_1 = 1$ and $v_3 = 2$.

Selberg (Lectures on Sieves) proved that

$$\left[\frac{R+1}{2}\right] \le v_R \le R.$$

"It would be of interest to compute v_R for a number of larger values (mine do not go beyond single digits) to see whether the ratio v_R/R approaches 1/2 or not."

Here are computations up to $R = 15$:

R	$\{r_1, r_2, \ldots, r_K\}$	v_R	$v_R/(R+1)$
1	$\{\,\}$	1	0.500
3	$\{3\}$	2	0.500
5	$\{3, 7\}$	3.117	0.520
7	$\{3, 6, 11\}$	4.143	0.518
9	$\{3, 6, 10, 14\}$	5.238	0.524
11	$\{3, 6, 9, 13, 18\}$	6.291	0.524
13	$\{3, 6, 9, 13, 17, 22\}$	7.309	0.522
15	$\{3, 6, 9, 12, 16, 20, 25\}$	8.337	0.521

We have extended these computations up to $R = 39$, and we found that

$$v_{39} = 20.575\ldots,$$

$$\frac{v_{39}}{40} = 0.515\ldots.$$

The End ...

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Thank you! ...