A Prime Number Theorem for Rankin-Selberg L-functions

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Introduction

•E and F denote finite Galois extensions of $\mathbb{Q}$ of degrees ℓ and ℓ', respectively.

• Let (π, V_π) (resp. (σ, V_σ)) be an automorphic cuspidal representation of $GL_m(\mathbb{A}_E)$ (resp. $GL_n(\mathbb{A}_F)$) with unitary central character.

• For any place lying over p denote by f_p, e_p (resp. f'_p, e'_p) the modular degree and ramification index of $E/\mathbb{Q}$ (resp. $F/\mathbb{Q}$) (these only depend on p since $E/\mathbb{Q}$ and $F/\mathbb{Q}$ are Galois).

• Denote by $\{\alpha_\pi(i,\nu)\}_{i=1,\dots,m}$ (resp. $\{\alpha_\sigma(j,\omega)\}_{j=1,\dots,n}$) the local parameters coming from the representation at the finite place ν of E (resp. ω of F).

• Denote by $\{\mu_\pi(i,\nu)\}_{i=1,\dots,m}$ (resp. $\{\mu_\sigma(j,\omega)\}_{j=1,\dots,n}$)the local parameters coming from the representation at the infinite place ν of E (resp. ω of F)

Classical Case

We will use the Rankin-Selberg L-functions $L(s, \pi \times \sigma)$ as developed by Jacquet, Piatetski-Shapiro, and Shalika [2], Shahidi [3], and Moeglin and Waldspurger [4]. The Rankin-Selberg L-function is defined as the infinite product

$$L(s, \pi \times \sigma) = \prod_{p \text{ prime}} \prod_{\nu | p} \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - \alpha_\pi(i, \nu) \alpha_\sigma(j, \nu) p^{-s})^{-1}$$

which converges absolutely for $Re(s) > 1$.(Jacquet and Shalika [5]). Now let

$$\frac{L'}{L}(s, \pi \times \sigma) = - \sum_{n \geq 1} \frac{\Lambda(n) a_{\pi \times \sigma}(n)}{n^s} \text{ for } Re(s) > 1$$

where $a_{\pi \times \sigma}(n) = f_p \sum_{\nu | p} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_\pi(i, \nu)^k \alpha_\sigma(j, \nu)^k$ for $n = p^k$, $f_p | k$, p prime.

Classical Case

By a prime number theorem for $L(s, \pi \times \tilde{\sigma})$ we mean the asymptotic behavior of the sum

$$\sum_{n \leq x} \Lambda(n) a_{\pi \times \tilde{\sigma}}(n) \tag{0.1}$$

Where $\tilde{\sigma}$ denotes the contragredient of σ. In this case, if one of π or σ is self-contragredient ($\pi \cong \tilde{\pi}$ or $\sigma \cong \tilde{\sigma}$) we have the following asymptotic formula for (0.1) (Ji, Gillespie [6]) due to Liu and Ye for $E = \mathbb{Q}$ [7]

$$= \begin{cases} \dfrac{x^{1+i\tau_0}}{1+i\tau_0} + O\{x \exp(-c\sqrt{\log x})\} \\ \qquad\qquad \text{if } \sigma \cong \pi \otimes |\det|^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}; \\ O\{x \exp(-c\sqrt{\log x})\} \\ \qquad\qquad \text{if } \sigma \ncong \pi \otimes |\det|^{it} \text{ for any } t \in \mathbb{R}. \end{cases}$$

for some constant $c > 0$.

Non-classical case

Suppose that E and F are cyclic extensions of prime degree ℓ and ℓ' respectively. Also suppose that $\pi^\gamma \cong \pi$ and $\sigma^{\gamma'} \cong \sigma$ where γ is a generator of $Gal(E/\mathbb{Q})$, and γ' is a generator of $Gal(F/\mathbb{Q})$. Here the Galois action is defined by $\pi^\gamma(g) = \pi(g^\gamma)$. By a result of Arthur and Clozel [1], π (resp. σ) is the base change lift of exactly ℓ (resp ℓ') non-equivalent cuspidal representations $\{\pi_{\mathbb{Q}} \otimes \eta^b_{E/\mathbb{Q}}\}_{b=0}^{\ell-1}$ (resp. $\{\pi_{\mathbb{Q}} \otimes \xi^b_{F/\mathbb{Q}}\}_{q=0}^{\ell'-1}$) of $GL_m(\mathbb{A}_{\mathbb{Q}})$ (resp. $GL_n(\mathbb{A}_{\mathbb{Q}})$) thus

$$L(s,\pi) = \prod_{b=0}^{\ell-1} L(s, \pi_{\mathbb{Q}} \otimes \eta^b_{E/\mathbb{Q}})$$

$$L(s,\sigma) = \prod_{q=0}^{\ell'-1} L(s, \sigma_{\mathbb{Q}} \otimes \xi^q_{F/\mathbb{Q}})$$

here $\eta_{E/\mathbb{Q}}$ and $\xi_{F/\mathbb{Q}}$ are idele class characters on $\mathbb{A}^\times_{\mathbb{Q}}$ associated to E and F by class field theory.

Define the Rankin-Selberg L-function over different fields $L(s, \pi \times_{E,F} \tilde{\sigma})$ by

$$L(s, \pi \times_{E,F} \tilde{\sigma}) := \prod_{b=0}^{\ell-1} \prod_{q=0}^{\ell'-1} L(s, \pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{b} \times \widetilde{\sigma_{\mathbb{Q}} \otimes \xi_{F/\mathbb{Q}}^{q}})$$

then for $n = p^k$ a prime power

$$a_{\pi \times_{E,F} \tilde{\sigma}}(n) = \sum_{b=0}^{\ell-1} \sum_{q=0}^{\ell'-1} a_{\pi_{\mathbb{Q}} \otimes \eta_{E/\mathbb{Q}}^{b}}(n) a_{\widetilde{\sigma_{\mathbb{Q}} \otimes \xi_{F/\mathbb{Q}}^{q}}}(n)$$

Non-classical case

Suppose that one of $\pi_{\mathbb{Q}}$ or $\sigma_{\mathbb{Q}}$ is self-contragredient, then

$$\sum_{n\geq 1} \Lambda(n) a_{\pi\times_{E,F}\tilde{\sigma}}(n)$$

$$= \begin{cases} \min\{\ell,\ell'\}\frac{x^{1+i\tau_0}}{1+i\tau_0} + O(x\exp(-c\sqrt{\log x})) \\ \text{if } \pi_{\mathbb{Q}}\otimes\eta^{i}_{E/\mathbb{Q}} \cong \xi^{j}_{F/\mathbb{Q}}\otimes|\det|^{i\tau_0} \text{ for some } i,j \text{ and } \tau_0\in\mathbb{R} \\ O(x\exp(-c\sqrt{\log x})) \\ \text{if } \pi_{\mathbb{Q}}\otimes\eta^{i}_{E/\mathbb{Q}} \not\cong \xi^{j}_{F/\mathbb{Q}}\otimes|\det|^{i\tau} \text{ for any } i,j \text{ and } \tau\in\mathbb{R} \end{cases}$$

for some constant $c > 0$.

Why the self-contragredient assumption?

In order to obtain the error term in the asymptotic formulas above we need classical results about zero free regions for $L(s, \pi \times \widetilde{\sigma})$. More specifically: $L(s, \pi \times \widetilde{\sigma})$ is non-zero in $Re(s) > 1$ (Shahidi [3]). Furthermore, if at least one of π or σ is self-contragredient, it is zero-free in the region

$$Re(s) \geq 1 - \frac{c_3}{\log(Q_{\pi \times \widetilde{\sigma}}(|t| + 2))}, \quad |t| \geq 1$$

and there is at most most one exceptional zero in the region

$$Re(s) \geq 1 - \frac{c_3}{\log(Q_{\pi \times \widetilde{\sigma}} c_4)}, \ |t| \leq 1$$

For some effectively computable constants c_3 and c_4 (Moreno [8], Sarnak [9], and Gelbart, Lapid, and Sarnak [10]). Here $Q_{\pi \times \widetilde{\sigma}}$ denotes the conductor.

• (Methods of Liu and Ye [7]) For the classical case:
Step 1: Prove a weighted version in the diagonal case

$$\sum_{n\leq x}(1-\frac{n}{x})\Lambda(n)a_{\pi\times\widetilde{\pi}}(n)=\frac{x}{2}+O(x\exp(-c\sqrt{\log x}))$$

More specifically using the formula

$$\frac{1}{2\pi i}\int_{(b)}\frac{y^s}{s(s+1)}ds=\begin{cases}1-1/y \text{ if } y\geq 1\\ 0 \text{ if } 0<y<1\end{cases}$$

taking $b=1+1/\log x$ we get

$$\sum_{n\leq x}(1-\frac{n}{x})\Lambda(n)a_{\pi\times\widetilde{\pi}}(n)=\frac{1}{2\pi i}\int_{(b)}J(s)\frac{x^s}{s(s+1)}ds$$

where $J(s)=-\sum_{n\geq 1}\frac{\Lambda(n)a_{\pi\times\widetilde{\pi}}(n)}{n^s}$.

Method of Proof

$$= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} J(s) \frac{x^s}{s(s+1)} ds + O(\frac{x}{T})$$

Now choose $-2 < a < -1$ and a large $T > 0$ (avoiding poles of gamma factors) and shift the contour to $Re(s) = a$ picking up residues along the way.

$$= \frac{1}{2\pi i} (\int_{C_1} + \int_{C_2} + \int_{C_3}) + Res\ J(s) \frac{x^s}{s(s+1)}$$

where

$$C_1 : b \geq Re(s) \geq a,\ \ t = -T; \quad (0.2)$$

$$C_2 : Re(s) = a,\ \ -T \leq t \leq T; \quad (0.3)$$

$$C_3 : a \leq Re(s) \leq b, t = T$$

Method of Proof

The three poles $s = 0, 1, -1$, some trivial zeroes and certain nontrivial zeroes will be passed by shifting the contour. For the residues corresponding to the trivial zeros it is enough to use the functional equation of the complete L-function

$$\Phi(s, \pi \times \widetilde{\pi}) = L(s, \pi) L_\infty(s, \pi) = \epsilon(s, \pi \times \widetilde{\pi}) \Phi(1 - s, \widetilde{\pi} \times \pi) \quad (0.4)$$

and the trivial bound $Re(\mu_{\pi \times \widetilde{\pi}}(i, j, \nu)) > -1$.
where

$$L_\infty(s, \pi \times \widetilde{\pi}) = \prod_{\nu | \infty} \prod_{i=1}^{m} \prod_{j=1}^{m} \Gamma_\nu(s + \mu_{\pi \times \widetilde{\pi}}(i, j, \nu))$$

and $\epsilon(s, \pi \times \widetilde{\pi}) = \tau(\pi \times \widetilde{\pi}) Q_{\pi \times \widetilde{\pi}}^{-s}$ with $Q_{\pi \times \widetilde{\pi}} > 0$ and $\tau(\pi \times \widetilde{\pi}) = \pm Q_{\pi \times \widetilde{\pi}}^{1/2}$. For the nontrivial zeroes we need the above zero free region.

For the integral over C_1 we use the fact that for any large $\tau > 0$ we can choose T in $\tau < T < \tau + 1$ so that whenever $-1 \leq \beta \leq 2$ then

$$J(\beta \pm iT) \ll \log^2(Q_{\pi \times \widetilde{\pi}})$$

thus

$$\int_{C_1} \ll \int_a^b \log^2(Q_{\pi \times \widetilde{\pi}} T) \frac{x^\beta}{T^2} d\beta \ll \frac{x \log^2(Q_{\pi \times \widetilde{\pi}} T)}{T^2}$$

and the same bound holds for the integral over C_3. For C_2 we need the fact that we can choose a so that whenever $|t| \leq T$, then $J(a + it) \ll 1$ so that

$$\int_{C_2} \ll \int_{-T}^{T} \frac{x^a}{(|t| + 1)^2} dt \ll \frac{1}{x}$$

Taking $T \gg \exp(\sqrt{\log x})$ the three integrals are $\ll x \exp(-c\sqrt{\log x})$.

Step2: Since the coefficients of the sum are nonnegative we can remove the weight $(1 - \frac{n}{x})$ using a classical method of de la Vallee Poussin to get

$$\sum_{n \le x} \Lambda(n) a_{\pi \times \widetilde{\pi}}(n) = x + O(x \exp(-c\sqrt{\log x}))$$

by considering

$$\int_1^x \sum_{n \le x} \Lambda(n) a_{\pi \times \widetilde{\pi}}(n) = \frac{x^2}{2} + O\left\{x^2 \exp(-c\sqrt{\log x})\right\}$$

thus

$$\frac{1}{h} \int_x^{x+h} \sum_{n \le x} \Lambda(n) a_{\pi \times \widetilde{\pi}}(n) = x + O\left\{x \exp(-\frac{c}{2}\sqrt{\log x}\right\} \qquad (0.5)$$

where $h = x \exp(-\frac{c}{2}\sqrt{\log x})$

$$\frac{1}{h} \int_{x-h}^{x} \sum_{n \le x} \Lambda(n) a_{\pi \times \widetilde{\pi}}(n) = x + O\left\{x \exp(-\frac{c}{2}\sqrt{\log x})\right\} \qquad (0.6)$$

and since $\sum_{n\le x}\Lambda(n)a_{\pi\times\widetilde{\pi}}(n)$ is and increasing function of x we get that it is bounded above by (0.5) and below by (0.6) so the result follows.

Step 3: Apply the following version of Perron's formula due to Liu Ye [7]

• Let $f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$ with abscissa of absolute convergence σ_a. Let $B(Re(s)) = \sum_{n \geq 1} \frac{|a_n|}{n^{Re(s)}}$. Then, for $b > \sigma_a$, $x \geq 2$, $T \geq 2$ we have

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds \tag{0.7}$$

$$+O\left\{ \sum_{x - x/\sqrt{T} < n \leq x + x/\sqrt{T}} |a_n| \right\} + O\left\{ \frac{x^b B(Re(s))}{\sqrt{T}} \right\}$$

So in our case

$$\sum_{n\le x}\Lambda(n)a_{\pi\times\widetilde{\sigma}}(n)=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\{-\frac{L'}{L}(s,\pi\times\widetilde{\sigma})\}\frac{x^s}{s}ds$$

$$+O\left\{\sum_{x-x/\sqrt{T}<n\le x+x/\sqrt{T}}|\Lambda(n)a_{\pi\times\widetilde{\sigma}}(n)|\right\}+O\left\{\frac{x^b\sum_{n\ge1}\frac{\Lambda(n)|a_{\pi\times\widetilde{\sigma}}(n)|}{n^b}}{\sqrt{T}}\right\} \tag{0.8}$$

where $b=1+1/\log x$ and $T\gg\exp(\sqrt{\log x})$.

• Step 4.

Now assuming π is self-contragredient but not necessarily σ, we need the following Tauberian theorem due to Ikehara [11]

• If f(s) is given for $Re(s) > 1$ by

$$f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$$

with $a_n \geq 0$, and if

$$g(s) = f(s) - \frac{1}{s-1}$$

has analytic continuation to $Re(s) \geq 1$, then

$$\sum_{n \leq x} a_n \backsim x$$

Using this we control both the error terms in Perron's formula, and proceed as in step 1 with the integral. □

By Cauchy's inequality

$$\sum_{x-x/\sqrt{T}<n\leq x+x/\sqrt{T}} |\Lambda(n)a_{\pi\times\widetilde{\sigma}}(n)| \tag{0.9}$$

$$\ll \left\{ \sum_{x-x/\sqrt{T}<n\leq x+x/\sqrt{T}} \Lambda(n)a_{\pi\times\widetilde{\pi}}(n) \right\}^{1/2} \tag{0.10}$$

$$\times \left\{ \sum_{x-x/\sqrt{T}<n\leq x+x/\sqrt{T}} \Lambda(n)a_{\sigma\times\widetilde{\sigma}}(n) \right\}^{1/2} \ll \sqrt{(\frac{x}{\sqrt{T}})(x)}$$

Method of Proof

- For the non-classical case we prove a lemma calculating the maximum number of twisted equivalent pairs using the fact that the representations are inequivalent and apply the previous result when $E = F = \mathbb{Q}$.

Original Hope: The Case $E \neq F$ in the large

Now let E and F be arbitrary finite Galois extensions, write

$$L(s,\pi) = \prod_{p \text{ prime}} \prod_{\nu | p} \prod_{i=1}^{m} \prod_{a=0}^{f_p - 1} (1 - \alpha_\pi(i,\nu)^{1/f_p} \omega_{f_p}^a p^{-s})^{-1}$$

where ω_{f_p} is a primitive f_p-th root of unity. Similarly

$$L(s,\sigma) = \prod_{p \text{ prime}} \prod_{\omega | p} \prod_{j=1}^{n} \prod_{b=0}^{f'_p - 1} (1 - \alpha_\sigma(j,\omega)^{1/f'_p} \omega_{f'_p}^b p^{-s})^{-1}$$

Define $L(s, \pi \times_{E,F} \sigma)$ by the formula

$$= \prod_{p} \prod_{\nu|p} \prod_{\omega|p} \prod_{i=1}^{m} \prod_{j=1}^{n} \prod_{a=0}^{f_p-1} \prod_{b=0}^{f'_p-1} (1 - \alpha_\pi(i,\nu)^{1/f_p} \alpha_\sigma(j,\omega)^{1/f'_p} \omega_{f_p}^a \omega_{f'_p}^b p^{-s})^{-1}$$

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