Zeros of Dirichlet series with periodic coefficients

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Zeta Functions, L-Functions and their Applications

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Motivation

General Riemann Hypothesis

Every $L_\chi(s)$ function, associated with a Dirichlet character χ, is zero-free in the open half-plane $\Re(s) > 1/2$.

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Question

What can we say about zeros in $\Re(s) > 1/2$ for Dirichlet series with periodic coefficients?

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Thus $\mathcal{F}$ is a $\mathcal{P}$-module in the set of all Dirichlet series.
- Let $\mathcal{D}^{\text{pr}}$ be the set of primitive Dirichlet characters.
- For ψ in $\mathcal{D}^{\text{pr}}$, we write
$$\mathcal{E}_\psi = \{P(s)L_\psi(s) \mid P \in \mathcal{P}\}.$$

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- Thus every Dirichelt series $\sum_{n \geq 1} \frac{a_n}{n^s}$, where $(a_n)_{n \geq 1}$ is a periodic sequence, can be written in a unique way as a finite sum

$$\sum_{\psi \in \mathcal{D}^{\mathrm{pr}}} P_\psi(s) L_\psi(s) \tag{1}$$

where the $P_\psi(s)$ are Dirichlet polynomials.

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  where the $P_\psi(s)$ are Dirichlet polynomials.
- Conversely, every expression of the form (1) is a Dirichlet series with periodic coefficients.

Remarks on Theorem 1

- Let's fix the period q. Codecà, Dvornicich, and Zannier (1998) showed that

$$\left\{ \chi\left(\frac{\cdot}{d}\right) \ : \ d|q \text{ and } \chi \text{ is a Dirichlet character mod } \frac{q}{d} \right\}$$

forms an orthogonal basis for the q-periodic sequences $(a_n)_{n\geq 1}$ with scalar product $\langle a, b\rangle = \sum_{n=1}^{q} a_n\overline{b_n}$.

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- Theorem 1 follows from from expressing this result in terms of primitive characters.

Definitions

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- We denote by $N_F(\sigma_1, \sigma_2, T)$ (respectively $N'_F(\sigma_1, \sigma_2, T)$) the number of zeros of the function $F(s)$ in the rectangle $\sigma_1 < \Re s < \sigma_2, |\Im s| \leq T$, counted with their multiplicities (resp. without their multiplicities).

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Then there exists a number $\eta = \eta(a) > 0$ *such that, for all real numbers* σ_1 *and* σ_2 *with* $1/2 < \sigma_1 < \sigma_2 \leq 1 + \eta$ *and all sufficiently large* T, *we have*

$$N_{F_a}(\sigma_1, \sigma_2, T) \asymp N'_{F_a}(\sigma_1, \sigma_2, T) \asymp T,$$

where the implied constants depend on a, σ_1, *and* σ_2.

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where the implied constants depend on a, σ_1, and σ_2.

Remark: When we write $F_a(s)$ according to Theorem 1 as

$$F_a(s) = \sum_{\psi \in \mathcal{D}^{\mathrm{pr}}} P_\psi(s) L_\psi(s)$$

the condition in Theorem 2 is equivalent to asking that $F_a(s)$ does not belong to one of the submodules generated by a single L_ψ.

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Let F be a Dirichlet series with periodic coefficients. The following are equivalent.

(i) $F(s)$ *does not vanish in* $\Re s > 1$.

(ii) $F(s) = P(s)L_\psi(s)$, *where* ψ *is a Dirichlet character and* $P(s)$ *is a Dirichlet polynomial that does not vanish in* $\Re s > 1$.

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Remark:

- The conditions $\Re s > 1$ can be replaced by $\Re s \geq 1$.
- If $\Re s > 1$ is replaced by $\Re s > 1/2$, the statement is equivalent to GRH.

Remarks on Theorem 2

- The upper bound $N_{F_a}(\sigma_1, \sigma_2, T) \ll T$ can be derived from an estimate of $\int_0^T |F_a(\sigma + it)|^2 dt$ due to Kačenas and Laurinčikas.

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- Both use the joint universal property for Dirichlet L-functions inside the critical strip.
- For $\Re(s) > 1$ we use Brouwer's fixed point theorem.

The Lower Bound

Let $\mathcal{C}$ be a finite set of at least two primitive Dirichlet characters, and let $(P_\psi)_{\psi\in\mathcal{C}}$ be a family of non-zero Dirichlet polynomials. Define

$$F(s) := \sum_{\psi\in\mathcal{C}} P_\psi(s) L_\psi(s).$$

Then there exists a number $\eta = \eta(F) > 0$ such that, for all real numbers σ_1 and σ_2 with $1/2 \le \sigma_1 < \sigma_2 \le 1 + \eta$ and all sufficiently large T, we have

$$N_F'(\sigma_1, \sigma_2, T) \gg_{F,\sigma_1,\sigma_2} T.$$

First Lemma for the Lower Bound in $\Re s > 1$

Let $D_n(R) := \{z = (z_j)_{1 \le j \le n} \in \mathbb{C}^n : |z_j| \le R \text{ for all } 1 \le j \le n\}$.

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Lemma 1.

Let q be a positive integer, and y and R be positive real numbers. Let $\chi_1, \ldots, \chi_n$ be pairwise distinct Dirichlet characters modulo q. Then there exists a real $\eta > 0$ such that for all fixed σ with $1 < \sigma \le 1 + \eta$, and for all prime numbers $p > y$, there exists a continuous function $t_p : D_n(R) \longrightarrow \mathbb{R}$, such that for all z in $D_n(R)$

$$z = \left(\sum_{p > y} \frac{\chi_j(p)}{p^{\sigma + i t_p(z)}} \right)_{1 \le j \le n}$$

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We can interpret this lemma as a linear system to be solved, where the unknowns are the infinite family of $\left(p^{-it_p}\right)_{p>y}$ that must be chosen in the unit circle, continuously in the parameter z.

Second Lemma for the Lower Bound in $\Re s > 1$

Lemma 2.
Let q and L be positive integers, and $R \geq 1$ be real. Let $\chi_1, \ldots, \chi_n$ be pairwise distinct Dirichlet characters modulo q. For all $1 \leq j \leq n$, let h_j be a non-zero rational function in L complex variables.

Second Lemma for the Lower Bound in $\Re s > 1$

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$$\left\{ z \in \mathbb{C}^n : \frac{1}{R} \leq |z_j| \leq R \right\} \subset$$
$$\left\{ \left(h_j\left(\frac{1}{p_1^{\sigma + it_{p_1}}}, \ldots, \frac{1}{p_L^{\sigma + it_{p_L}}} \right) \prod_{p > p_L} \left(1 - \frac{\chi_j(p)}{p^{\sigma + it_p}} \right)^{-1} \right)_{1 \leq j \leq n} : t_p \in \mathbb{R} \right\}$$

Brouwer's Fixed Point Theorem is used to prove Lemma 2

The first Lemma shows there are continuous functions t_p such that

$$w_j = \sum_{p>y} \frac{\chi_j(p)}{p^{\sigma + it_p(w)}}, \qquad w \in D_n(1+R'),\ 1 \le j \le n,$$

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where $R' = \pi + \log R$. Define the error term E by

$$\left(\sum_{p>y} \log\left(1 - \frac{\chi_j(p)}{p^{\sigma+it_p}}\right)\right)_{1\le j\le n} = \left(- \sum_{p>y} \frac{\chi_j(p)}{p^{\sigma+it_p}}\right)_{1\le j\le n} + E\left((t_p)_{p>y}\right).$$

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Now $|E_j\left((t_p)_{p>y}\right)| \leq \sum_p \frac{1}{p^2} < 1$. Let $z \in D_n(R')$ be fixed. Thus

$$F:\ D_n(1+R') \longrightarrow D_n(1+R'), \quad w \longmapsto z + E\left((t_p(w))_{p>y}\right)$$

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The Brouwer fixed point theorem shows $\exists w \in D_n(1+R')$ with

$$\left(-\sum_{p>y} \log\left(1 - \frac{\chi_j(p)}{p^{\sigma+it_p(w)}}\right)\right)_{1\le j\le n} = z.$$

Proof of Lemma 1: Change of Variables

Assume $n = \varphi(q)$. We have

$$\sum_{p>y} \frac{\chi_j(p)}{p^{\sigma+it_p}} = \sum_{\substack{1\le a\le q\\(a,q)=1}} \chi_j(a) \sum_{\substack{p>y\\p\equiv a\,(q)}} \frac{1}{p^{\sigma+it_p}} \tag{2}$$

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To change variables we write $z = Cw$, where

$$C := (\chi_j(a))_{\substack{1\le a\le q,\ (a,q)=1 \\ 1\le j\le \varphi(q)}}, \quad \theta_p = -(\log p)(t_p \circ C).$$

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To prove the lemma, it is sufficient to solve the system

$$\sum_{\substack{p>y\\p\equiv a\,(q)}} \frac{e^{i\theta_p}}{p^{\sigma}} = w_a, \qquad 1 \le a \le q,\ (a,q) = 1, \tag{3}$$

in $(\theta_p)_{p>y}$, continuously in $w \in D_{\varphi(q)}(\|C^{-1}\|_\infty R)$.

Proof of Lemma 1: Choosing angles continuously

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$$\frac{1}{3} \approx \lambda_0 := \frac{1}{S_a} \sum_{\substack{y<p\leq p_{1,a} \\ p\equiv a\,(q)}} \frac{1}{p^\sigma}, \quad \frac{1}{3} \approx \lambda_1 := \frac{1}{S_a} \sum_{\substack{p_{1,a}<p\leq p_{2,a} \\ p\equiv a\,(q)}} \frac{1}{p^\sigma}$$

and write $\lambda_2 := 1 - \lambda_0 + \lambda_1$.

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and write $\lambda_2 := 1 - \lambda_0 + \lambda_1$. We choose

$$\theta_p = \begin{vmatrix} 0 & \text{if } y < p \leq p_{1,a} \\ \pi + u_1 & \text{if } p_{1,a} < p \leq p_{2,a} \\ \pi - u_2 & \text{if } p_{2,a} < p \end{vmatrix}$$

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It is sufficient to solve, in the real unknowns u_1 and u_2, continuously in w_a for $|w_a| \leq \|C^{-1}\|_\infty R$, the equation

$$\lambda_1 e^{iu_1} + \lambda_2 e^{-iu_2} = \lambda_0 - \frac{w_a}{S_a} \tag{4}$$

The image under the diffeomorphism F

$$F : \left]0, \frac{\pi}{2}\right[^2 \longrightarrow \mathbb{C}, \quad (u_1, u_2) \longmapsto \lambda_1 e^{iu_1} + \lambda_2 e^{-iu_2}.$$

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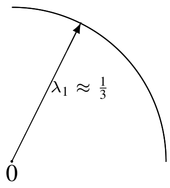

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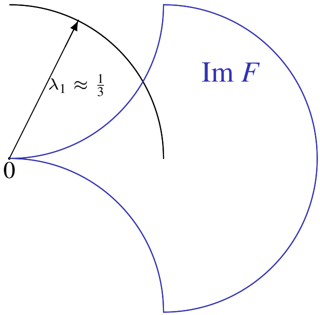

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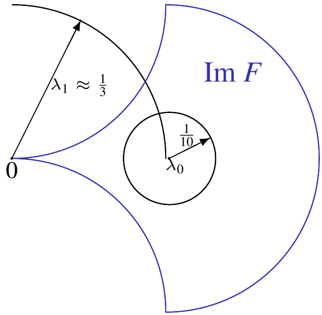

Figure: The image of F, depicted by the region with the blue boundary, contains the disk with center λ_0 and radius $\frac{1}{10}$.

Proof of Lower Bound: Preparation

If $\sigma_1 < 1$ then $N_F'(\sigma_1, \sigma_2, T) \gg_{F,\sigma_1,\sigma_2} T$ by a result of Kaczorowski and Kulas. We may thus assume $\sigma_1 \geq 1$.

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We choose q to be the least common multiple of the conductors of the $\psi \in \mathcal{C} = \{\psi_1, \ldots, \psi_n\}$ with $2 \leq n \leq \varphi(q)$. We use the notation

$$F_j(s) = P_{\psi_j}(s) L_{\psi_j}(s), \quad P_{\psi_j}(s) = \sum_{k \geq 1} \frac{c_{j,k}}{k^s}$$

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We choose $y = p_L$ such that if p divides a k for which there is a j such that $c_{j,k} \neq 0$, then $p \leq y$. Denoting by χ_j the Dirichlet character modulo q that is induced by ψ_j we can thus write

$$F_j(s) = h_j\left(\frac{1}{p_1^s}, \dots, \frac{1}{p_L^s}\right) \prod_{p > p_L} \left(1 - \frac{\chi_j(p)}{p^s}\right)^{-1}$$

where h_j is a rational function, not identically equal to zero, with no poles in $\{(z_1, \dots, z_L) \in \mathbb{C}^L : |z_l| < 1\}$.

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We write

$$G_j(s) := h_j\left(\frac{1}{p_1^{s + it_{p_1}}}, \dots, \frac{1}{p_L^{s + it_{p_L}}}\right) \prod_{p > p_L} \left(1 - \frac{\chi_j(p)}{p^{s + it_p}}\right)^{-1}.$$

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As $n \geq 2$, we have $\sum_{j=1}^{n} G_j(\sigma) = 0$.

The Lower Bound: Truncating the Products

We now choose a circle $C = C(\sigma, r)$ centered at $\sigma = \frac{\sigma_1+\sigma_2}{2}$ and with a radius r with $0 < r < \frac{\sigma_2-\sigma_1}{2}$, such that $\sum_{j=1}^{n} G_j(s)$ does not vanish on C. We write

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We can choose a prime number $p_M \geq p_L$ such that for all j, $1 \leq j \leq n$,

$$\left| F_j(z) - h_j\left(\frac{1}{p_1^z}, \ldots, \frac{1}{p_L^z}\right) \prod_{p_L < p \leq p_M} \left(1 - \frac{\chi_j(p)}{p^z}\right)^{-1} \right| < \frac{\gamma}{3n}, \quad \Re z \geq \sigma - r,$$

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The Lower Bound: Weyl's Criterion and Rouche's Theorem

By Weyl's criterion, we know that the set $\{p_1^{it}, \ldots, p_M^{it}\}$ is uniformly distributed in $\{z : |z| = 1\}^M$.

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$$\max_{s \in C} \left| \sum_{j=1}^{n} F_j(s+it) - G_j(s) \right| < \gamma = \min_{s \in C} \left| \sum_{j=1}^{n} G_j(s) \right|$$

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As $\sum_{j=1}^n G_j(\sigma) = 0$, it follows by Rouche's theorem that $F(s+it) = \sum_{j=1}^n F_j(s+it)$ has at least one zero in $|s-\sigma| < r$.

The Upper Bound

Let $a = (a_n)_{n \geq 1}$ be a non-zero periodic sequence. Then

$$N_{F_a}\left(\frac{1}{2} + u, +\infty, T\right) \ll_a T\frac{\log(1/u)}{u}$$

for $0 < u \leq 1/2$ and $T \geq 1$.

Derivation of the upper bound

Use Littlewood's lemma together with the following estimate from Kačenas and Laurinčikas: For $1/2 < \sigma < 1$,

$$\int_0^T |F_a(\sigma + it)|^2 dt = \frac{T}{q^{2\sigma}} \sum_{j=1}^{q} |a_j|^2 \zeta(2\sigma, j/q) + O\left(\frac{q^{2-2\sigma} T^{2-2\sigma} \sum_{j=1}^{q} |a_j|^2}{(2\sigma - 1)(1-\sigma)} \right)$$
$$= O_a\left(\frac{T}{(2\sigma - 1)(1-\sigma)} \right),$$

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$$= O_a\left(\frac{T}{(2\sigma - 1)(1 - \sigma)} \right),$$

where $\zeta(s, r)$ is the Hurwitz zeta function. By Jensen's inequality,

$$\int_0^T \log |F_a(\sigma+it)| dt \leq \frac{T}{2} \log \left(\frac{1}{T} \int_0^T |F_a(\sigma + it)|^2 dt \right) = O_a(T \log(1/u)),$$

for $\sigma = (1 + u)/2$.

Summary

- Every Dirichlet series $F_a(s)$ with periodic coefficients can be written uniquely in the form

$$F_a(s) = \sum_{\psi \in \mathcal{D}^{\mathrm{pr}}} P_\psi(s) L_\psi(s)$$

where the $P_\psi(s)$ are Dirichlet polynomials.

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  where the $P_\psi(s)$ are Dirichlet polynomials.
- If the above sum has more than one non-zero term, then there exists a number $\eta = \eta(a) > 0$ such that, for all real numbers σ_1 and σ_2 with $1/2 < \sigma_1 < \sigma_2 \leq 1 + \eta$ and all sufficiently large T, we have

$$N_{F_a}(\sigma_1, \sigma_2, T) \asymp_{a,\sigma_1,\sigma_2} N'_{F_a}(\sigma_1, \sigma_2, T) \asymp_{a,\sigma_1,\sigma_2} T.$$

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