Bounds for the coefficients of powers of the Δ-function

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The Δ-function

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- Its coefficients

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q - 24q^2 + 252q^3 + \cdots$$

satisfy remarkable properties.

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$$|\tau(p)| \leq 2p^{11/2}.$$

- The last inequality follows from Deligne's proof of the Weil conjectures, and implies that $|\tau(n)| \leq d(n)n^{11/2}$.

Modular forms facts

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- This vector space is finite dimensional, and for each $n \geq 1$, there is a canonical linear operator T_n that acts on S_k.
- There is a basis for S_k consisting of forms that are simultaneous eigenfunctions for all these operators.
- For these forms (normalized so $a(1) = 1$), $|a(n)| \leq d(n)n^{(k-1)/2}$.

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- If $k \geq 1$, write

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- Q: How large is $\tau_k(n)$ as a function of k and n?
- A: There is a constant C_k so that

$$|\tau_k(n)| \leq C_k d(n) n^{(12k-1)/2}.$$

The constant C_k

- This is because we can write

$$\Delta^k = \sum_{i=1}^{k} c_i f_i,$$

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- If $C_k = \sum_{i=1}^{k} |c_i|$, Deligne's bound gives the result we want.
- How large is C_k as a function of k?

Numerical data

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Main theorem

Theorem (R, 2008)

For $k > 1$, we have

$$\log(C_k) = -6k\log(k) + 6k\log\left(\frac{2\pi^3 e}{27\Gamma(2/3)^6}\right) + O(\log(k)).$$

- If f and g are two cusp forms of weight k, define the *Petersson inner product* of f and g to be

$$\langle f, g\rangle = \frac{3}{\pi}\int_{\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})} f(x+iy)\overline{g(x+iy)}y^k \frac{dx\,dy}{y^2}.$$

Overview of proof (1/3)

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- If f_i and f_j are two distinct Hecke eigenforms, then $\langle f_i, f_j \rangle = 0$.

- Suppose we have explicit bounds

$$B_1 \leq \langle f_i, f_i \rangle \leq B_2$$

on the Petersson norms of the Hecke eigenforms.

Overview of proof (2/3)

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- If we write

$$\Delta^k = \sum_{i=1}^{k} c_i f_i,$$

we get

$$\langle \Delta^k, \Delta^k \rangle = \sum_{i=1}^{k} |c_i|^2 \langle f_i, f_i \rangle.$$

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- This gives

$$\frac{\langle \Delta^k, \Delta^k \rangle}{B_2} \leq \sum_{i=1}^{k} |c_i|^2 \leq \frac{\langle \Delta^k, \Delta^k \rangle}{B_1}.$$

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- Applying the Schwarz inequality gives bounds on $C_k = \sum_{i=1}^{k} |c_i|$.
- It suffices to compute bounds on $\langle \Delta^k, \Delta^k \rangle$ and $\langle f_i, f_i \rangle$.

Bounds on $\langle \Delta^k, \Delta^k \rangle$

- Elementary arguments give that

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- Here $B = \left(\frac{\sqrt{2}\pi}{3\Gamma(2/3)^3} \right)^{24}$.

L-functions

- If f_i is a Hecke eigenform of weight $12k$, then

$$L(\mathrm{Sym}^2 f_i, 1) = \frac{\pi^2}{6} \cdot \frac{(4\pi)^{12k} \langle f_i, f_i \rangle}{(12k-1)!}.$$

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- Then,

$$L(\mathrm{Sym}^2 f_i, s) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - \alpha_p^{-2} p^{-s})^{-1}.$$

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- Lower bounds for L-functions at $s = 1$ are in general difficult and are equivalent to the problem of zeroes close to $s = 1$.
- In this case, work of Goldfeld, Hoffstein, and Lieman solves the problem.

No Siegel zeroes

Lemma

If f_i is a Hecke eigenform of weight $12k$, then

$$L(\mathrm{Sym}^2 f_i, s) \neq 0$$

for $s > 1 - \frac{5-2\sqrt{6}}{10\log(12k)}$.

Proof of Lemma

- Let

$$L(\mathrm{Sym}^4 f_i, s) = \prod_p (1 - \alpha_p^4 p^{-s})^{-1} (1 - \alpha_p^2 p^{-s})^{-1} (1 - p^{-s})^{-1} \cdot (1 - \alpha_p^{-2})^{-1} (1 - \alpha_p^{-4} p^{-s})^{-1}.$$

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- Let
$$L(s) = \zeta(s)^2 L(\mathrm{Sym}^2 f_i, s)^3 L(\mathrm{Sym}^4 f_i, s).$$

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- This function has a double pole at $s = 1$, a triple zero at any zero of $L(\mathrm{Sym}^2 f_i, s)$ and non-negative Dirichlet coefficients.
- A standard argument shows that $L(\mathrm{Sym}^2 f_i, s)$ cannot have a zero too close to $s = 1$.

Lower bound on $\langle f_i, f_i \rangle$

Lemma

If f_i is a Hecke eigenform of weight $12k$, then

$$L(\mathrm{Sym}^2 f_i, 1) > \frac{1}{64 \log(12k)}.$$

Proof of Lemma (1/3)

- Let

$$L(f_i \otimes f_i, s) = \zeta(s) L(\mathrm{Sym}^2 f_i, s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

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- Let β be a real zero of $L(\mathrm{Sym}^2 f, s)$ and define
$$I = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{L(f_i \otimes f_i, s+\beta) x^s \, ds}{s \prod_{r=2}^{10} (s+r)}.$$

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- The bounds $a(n) \geq 0$ and $a(n^2) \geq 1$ give

$$I \geq 4.53 \cdot 10^{-7}.$$

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- The other two residues are negative. This gives

$$I - \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{L(f_i \otimes f_i, s+\beta) x^s \, ds}{s\prod_{r=2}^{10}(s+r)} \leq \frac{L(\mathrm{Sym}^2 f_i, 1) x^{1-\beta}}{(1-\beta)\prod_{r=2}^{10}(1-\beta+r)}.$$

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- Plugging in the result of the previous lemma gives the desired result.
- Relating $L(\mathrm{Sym}^2 f_i, 1)$ with $\langle f_i, f_i \rangle$ gives explicit lower bounds on the Petersson norm.
- Upper bounds on $\langle f_i, f_i \rangle$ can be derived using standard arguments.

Remaining questions

Is the constant $C_k = \sum_{i=1}^{k} |c_i|$ optimal in the inequality

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Conjecture

We have

$$C_k = \sup_{n \geq 1} \frac{|\tau_k(n)|}{d(n) n^{(12k-1)/2}}.$$