Special values of L-functions

Ameya Pitale

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Integers for which neither $\zeta_\infty(s)$ nor $\zeta_\infty(1-s)$ has a pole are precisely even positive integers.

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For $p \nmid N$ $\qquad L_p(s,f) = \dfrac{1}{1 - a_p p^{-s} + \chi(p) p^{k-1} p^{-2s}}$

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Theorem (Shimura, '77)

Let $f \in S_k(N,\chi)$ be a primitive form. Then there are two complex numbers $c_\pm(f)$ such that, for any integer $1 \le m \le k-1$, we have

$$\frac{L(m,f)}{\pi^m c_\pm(f)} \in \mathbb{Q}(f,\chi) \subset \overline{\mathbb{Q}}.$$

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Integers for which neither $L_\infty(s,f)$ nor $L_\infty(k-s,f)$ have a pole are $\{1,2,\cdots,k-1\}$.

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Let M be a "motive" and $L(s, M)$ be the motivic L-function. Then there exist two complex numbers $c_{\pm}(M)$ such that for any "critical point" m we have

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Shimura, Garrett, Harris, Harder, Clozel, Mahnkopf, Blasius, Raghuram .....

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$$\mathrm{GSp}_4 := \{g \in \mathrm{GL}_4 : {}^t g J g = \mu(g) J, \mu(g) \in \mathrm{GL}_1, J = \begin{bmatrix} & \mathbf{1}_2 \\ -\mathbf{1}_2 & \end{bmatrix}\}$$

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Siegel cusp form: $S_l^{(2)}(\mathrm{Sp}_4(\mathbb{Z}))$ space of holomorphic functions $F : \mathfrak{h}_2 \to \mathbb{C}$ satisfying

$$F((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^l F(Z) \text{ for all } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_4(\mathbb{Z}).$$

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where, for almost all p

$$L_p(s, \pi_p \times \tau_p) = \frac{1}{\text{degree 8 polynomial in } p^{-s}}$$

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Deligne's conjecture on special values.

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Can you find functions Φ_p, Ψ_p such that

$$Z(s, \pi_p, \tau_p) = L_p(s, \pi_p \times \tau_p)$$

Integral representation of L-function

Theorem (P.-Schmidt, '08-'09)

Let $F \in S_l^{(2)}(\mathrm{Sp}_4(\mathbb{Z}))$ *and* τ *be any cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. *Then it is possible to choose functions* Φ_p *and* Ψ_p *such that*

$$Z(s, \pi_F, \tau) = L(s, \pi_F \times \tau).$$

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Theorem (P.-Schmidt, '08-'09)

Let $F \in S_l^{(2)}(\mathrm{Sp}_4(\mathbb{Z}))$ *and* $f \in S_l(N, \chi)$ *for any* N, χ. *Then*

$$\frac{L(\frac{l}{2} - 1, \pi_F \times \tau_f)}{\pi^{5l-8}\langle F, F\rangle\langle f, f\rangle g(\chi)} \in \overline{\mathbb{Q}}.$$