ANALYTIC EXPERIMENTS BEYOND ENDOSCOPY

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1. Motivation

1.1. **The Functoriality conjecture.** The project I wish to describe is in the context of the so-called Beyond Endoscopy proposal of R. Langlands in 2001 [8], which offers a way of attacking Langlands' Functoriality Conjecture, posed some fifty years ago. The conjecture goes roughly as follows:

Given a groups G that is *reductive*—which for us will mean something close to a matrix group)—one can associate its L-group, denoted ${}^LG$. Now given two such groups H, G with a homomorphism

$$ {}^LH \xrightarrow{\psi} {}^LG $$

Langlands conjectures that there should be an associated transfer of *automorphic forms* on H to automorphic forms on G

$$ \Pi(H) \to \Pi(G), $$

where an automorphic form on G is a function from G to $\mathbb{C}$ satisfying certain properties. At this level of generality, it is hard to see why such a conjecture could be difficult or important; in fact, a closer look will show that it is in fact both: many classical problems in number theory can be phrased in terms of automorphic forms, and proving transfers as above in special cases is often progress on these problems and their generalizations.

An important tool that has been used to solve many specific cases is the *trace formula*. The case where H is an endoscopy group of G is one such case, but we will not need to know the details of this to attempt on certain cases *beyond* endoscopy. In particular, for us, we will consider only $G = GL(n, \mathbf{Q})$, in which case ${}^LG = GL(n, \mathbf{C}) \times \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$.

1.2. **The Beyond Endoscopy idea.** The proposal of Langlands now goes as follows: given an automorphic form π on G and a representation r of ${}^LG$, one can define an automorphic L-function

$$ L(s, \pi, r) = \sum_{n=1}^{\infty} \frac{a_{\pi,r}(n)}{n^s}, $$

where the denominators $a_{\pi,r}(n)$ are complex numbers defined by π and r; this includes as special cases the classical zeta functions and L-functions of Riemann, Dedekind, Dirichlet, Artin, and Hecke. It is expected that if π is a transfer of an automorphic form π_H on a smaller group H, then there is an equality of L-functions

$$ L(s, \pi, r) = L(s, \pi_H, r \circ \psi) $$

and $L(s, \pi, r)$ has a pole of order $m_\pi(r)$ at $s = 1$. Here is the catch: in general we don't even know that the L-function is even defined at $s = 1$! So following

Langlands' suggestion, we define the order of the pole in a roundabout way: if the function were defined at (and analytic in a neighborhood of) $s = 1$, its residue there would be given as

$$\mathrm{Res}_{s=1}L(s,\pi,r) = \lim_{X\to\infty}\frac{1}{X}\sum_{n<X} a_{\pi,r}(n).$$

This expression is nonzero if and only if there is no pole at $s = 1$, so we'll take the expression on the RHS as our substitute definition of $m_\pi(r)$, not knowing that the limit actually exists. In theory, one would want to see something like

$$\sum_{n<X} a_{\pi,r}(n) = O(X^{1-\epsilon}),$$

for some $\epsilon > 0$, though as P. Sarnak observes, this is already a 'quasi-Riemann hypothesis' for the L-function in question.

Remark 1.1. Langlands really uses the residue of the logarithmic derivative of $L(s,\pi,r)$, which gives exactly the order of the pole, but following an observation of Sarnak the sum over primes which this results in is harder to analyze than the sum over integers that we get as above.

Now the Arthur-Selberg trace formula of G is an identity of the shape:

$$\sum_\pi \mathrm{tr}(\pi(f)) + (\mathrm{Cont}) = \sum_\gamma \int_G f(g^{-1}\gamma g)dg$$

where f is a smooth, compactly supported function on G, and $\mathrm{tr}(\pi(f))$ are traces of certain operators, which for moment we'll think of as generalizations of traces of matrices. The sum on the RHS is over conjugacy classes of G, thus referred to as the geometric side, while the sum over automorphic forms on G on the LHS is called the spectral side. We defer the precise definition of the term (Cont).

The new idea is to write down an expression that only detects functorial lifts, and allows the use of the trace formula, since interchanging summations we get:

$$\begin{aligned}\sum_\pi m_\pi(r)\mathrm{tr}(\pi(f)) &= \sum_\pi \Big(\lim_{X\to\infty}\frac{1}{X}\sum_{n<X} a_{\pi,r}(n)\Big)\mathrm{tr}(\pi(f)) \\ &= \lim_{X\to\infty}\frac{1}{X}\sum_{n<X}\sum_\pi a_{\pi,r}(n)\mathrm{tr}(\pi(f))\end{aligned}$$

The goal is to show that this limit exists, and has nice estimates. The inner sum is itself a trace formula, so we may apply the identity to get

$$\lim_{X\to\infty}\frac{1}{X}\sum_{n<X}\Big(\sum_\gamma \int_G f(g^{-1}\gamma g)dg - (\mathrm{Cont})\Big). \tag{1}$$

This is the sum we would like to analyze.

1.3. **Previous work.** Langlands considers the case where G is $GL(2)$ and r is the symmetric k-th power Sym^k, thus:

$${}^L GL(2) \xrightarrow{\mathrm{Sym}^k} {}^L GL(k+1)$$

where for $k = 1$ one has simply the standard representation of $GL(n)$ acting on $\mathbf{C}^n$, in which case there is no expectation of poles on the limit should be zero. This analysis was initiated by Langlands, and completed in the 2012 thesis of A. Altuğ

[1]. The case $k = 2$ was carried out by A. Venkatesh in his 2002 thesis [11], using a variant of the trace formula, that is the Kuznetsov trace formula. In this case $L(s, \pi, \mathrm{Sym}^2)$ has a pole when π is a grössencharacter of a quadratic extension. The analytic difficulty of carrying out the same analysis for higher k is discussed by Sarnak [**?**] and also Venkatesh [11]; it is that when k is large the lengths of exponential of exponential sums considered become too short to be complete sums, thus difficult to manipulate.

Partial attempts of this method for other representations has been carried out independently by P.E. Hermann and D. White using the Kuznetsov trace formula and its generalization—the relative trace formula.

2. Background

2.1. **The basics.** For what we would like to carry out, the basic background you should have is most of the following:

(1) Basic complex analysis: what is a pole, residue, analytic continuation, Poisson summation etc. For example, `http://www.math.umn.edu/~garrett/m/complex/`.
(2) Basic linear algebra: Jordan decomposition, characteristic polynomials, etc.
(3) Group theory: what is a centralizer, conjugacy class, orbits etc.
(4) Simple modular arithmetic: see (3) below on character sums.

For (2) and (3) we will mostly only need this in the context of the matrix group $GL(3)$.

Plus, familiarity with any L-function or zeta function would be helpful (`http://www.math.umn.edu/~garrett/m/mfms/`). This will lead into our particular tools:

(1) Approximate functional equation. See, for example, §10.4 of [6]
(2) Character sums, particularly Kloosterman sums. Again, §12 of [6]
(3) Class number formula. Many, many references on this will appear in an internet search for '`class number formula`'. NB: its *proof* will not be particularly important to us, rather the result itself as applied below.

The papers we will rely heavily on will be [8], [1], and [9]. In general the book [6] will be a useful reference for tools that we expect to use.

2.2. **Beyond the basics.** For a better optic on what we are trying to do, you should know

(1) The Arthur-Selberg trace formula: `http://www.claymath.org/library/cw/arthur/pdf/62.pdf`, and more generally `http://www.claymath.org/publications/collected-works-james-g-arthur`.
(2) The adèlic language: while many expositions on this is available online, J. Tate's original thesis is the best, c.f. `https://en.wikipedia.org/wiki/Tate's_thesis`, which does not seem to be online.
(3) The Langlands Program in general, what are its goals and origins. A gentle introduction is here: `http://www.ams.org/journals/bull/1984-10-02/S0273-0979-1984-15237-6/S0273-0979-1984-15237-6.pdf`.

We want to study an integral which is a product of an integral over $\mathbf{R}$ and integrals over $\mathbf{Q}_p$. I plan to make progress on the $\mathbf{Q}_p$ side so that there is an explicit expression ready to be used as in [8], and you would get to blackbox this as does [1]. Our goal would be to take the $\mathbf{R}$ integral, make it explicit, then find a way to apply the approximate functional equation and Poisson summation at the right moment. See 3.1.2 below for details.

3. Projects

Following the solution of the Fundamental Lemma—which solves the problem of endoscopy—in 2009, there has been renewed interest in methods Beyond Endoscopy. In particular, recent work of Frenkel-Langlands-Ngô [4] and J. Arthur [2] are in the context of the Arthur-Selberg trace formula, rather than that of Kuznetsov. In the problems outlined below, I take the paradigm of the former, and as with work of Altuğ and Venkatesh, restrict to working over the real numbers, which by and large avoids the adélic formulation.

3.1. **Standard representation for GL(3).** As mentioned above, the simplest case of the standard representation for $GL(2)$ was initiated by Langlands and completed by Altuğ. The most difficult part of the analysis involves the elliptic conjugacy classes of the geometric sum in (1); it takes the form

$$\sum_{\pm}\sum_{r}\theta^{\pm}_{\infty}\left(\frac{r}{2p^{k/2}}\right)\sum_{f}\frac{1}{f}\,L\Big(1,\Big(\frac{(r^2\pm 4p^k)/f^2}{\cdot}\Big)\Big) \tag{2}$$

where the function θ_∞ is the (archimedean) orbital integral, and the $L(1,-)$ is related by the Class Number Formula to the volume of the centralizer of γ in G, and also the p-adic orbital integrals. Altuğ then uses (i) an approximate functional equation to the f sum, viewed as a Dirichlet series, to interchange the order of summation, and then (ii) the Poisson Summation formula to the inner sum. The result is then analyzed to produce the expected cancellation.

The analysis of Langlands and Altuğ provides a framework of how to execute the Beyond Endoscopy method, and from studying them Arthur has recently proposed a list of problems to be considered, some more approachable than others. These obstructions might be clarified through the following

Problem 1. *Carry out the method for $GL(3)$ and the standard representation.*

The geometric side of the trace formula for $GL(3)$ has been studied by Flicker in the context of base change, and more recently in the 2011 thesis of J. Matz [9] for noncompactly supported test functions. In the rest of the section we setup the problem in greater detail.

3.1.1. *Indexing the sums.* Two elements of $GL(3)$ are contained in the same conjugacy class if and only if they have the same characteristic polynomial. We call an element γ of $GL(3,\mathbf{Q})$ *semisimple* if it has distinct eigenvalues, and furthermore *elliptic* if its characteristic polynomial is irreducible. By the Jordan decomposition, any element γ can be factored into its semisimple and unipotent parts.

We divide the conjugacy classes into characteristic polynomials by the number of irreducible factors:

(1) 1 irredcuible factor: these are regular elliptic elements, in bijection with classes of cubic extensions of $\mathbf{Q}$
(2) 2 irreducible factors: these are only semisimple classes, corresponding to classes of quadratic extensions of $\mathbf{Q}$
(3) 3 irreducible factors
    (a) No repeated roots, so γ is semisimple with distinct eigenvalues in $\mathbf{Q}$.

(b) Two repeated roots, hence in bijection with distinct pairs in $\mathbf{Q}^\times \times \mathbf{Q}^\times$, dividing further into the forms

$$\begin{pmatrix} x & & \\ & x & \\ & & y \end{pmatrix}, \begin{pmatrix} x & x & \\ & x & \\ & & y \end{pmatrix}$$

which we will refer to as central and regular unipotent classes of $GL(2)$.

(c) Three repeated roots, hence in bijection with $\mathbf{Q}^\times$, dividing further into the forms x times

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$$

which we will refer to as central, minimal unipotent, and regular unipotent classes of $GL(3)$.

Weighted orbital integrals associated to these elements were computed by Flicker [3] and Matz. What we would like is to specialize these computations to obtain explicit expressions using the test function similar to that of Langlands.

3.1.2. *Regular elliptic terms.* The regular elliptic orbital integrals look initially like

$$\sum_{\gamma \text{ ell}} \mathrm{vol}(Z_{\mathbf{Q}} G_\gamma(\mathbf{Q}) \backslash G_\gamma(\mathbf{A})) \int_{G_\gamma(\mathbf{A}) \backslash G(\mathbf{A})} f(g^{-1}\gamma g) dg$$

where $\mathbf{A}$ denotes the adèle ring of $\mathbf{Q}$. To analyze these terms, we follow the setup of Altuğ: the volume factors, again by the class number formula, can be described by the value at $s = 1$ of Hecke L-functions associated to cubic characters. One difficulty here is the formula is no longer as simple as in the case of $GL(2)$, in our case one has

$$\mathrm{Res}_{s=1}\zeta_K(s) = L(1,\chi)L(1,\chi^2) = \frac{2^{r_1}(2\pi^{r_2})hR}{\sqrt{|D|}}$$

where χ is a character of a cubic extension K of $\mathbf{Q}$. In particular, now we encounter a *product* of L-functions, which will be a difficulty in introducing an approximate functional equation.

The orbital integral, on the other hand, will have singularities which one hopes can be surmounted with the help of the approximate functional equation for said L-functions. In work of Flicker he shows that Poisson summation formula can still be applied to singular orbital integrals with functions that not smooth but regular, which may be of use here.

Project 1. *Write an explicit expression for the regular elliptic integrals for $GL(3)$ analogous to (2), involving (i) the volume factors, (2) archimedean orbital integral θ_∞, and (3) p-adic orbital integrals.*

Note that the computations of p-adic orbital integrals for $GL(3)$ are contained in [9, p.127], for a noncompactly supported function, see also the computations in Lemma 1 and §2.5 in [8]

Succeeding this, we will be in a position to apply the Poisson summation formula. This step is natural in the sense that the summation formula is commonly used in the trace formula, for example to obtain an invariant trace formula, but more importantly, in every example of beyond endoscopy worked out so far. Indeed, Flicker comments that appealing to the Poisson summation formula at the right moment is fundamental for applications.

Once the summation formula is applied, we would like to see the dominant term $\hat{f}(0)$ in the resulting sum to cancel with the contribution of the trivial representation $\mathrm{tr}(\mathbf{1}(f))$ to the spectral side of the trace formula, as was shown in [4] and [1]. In summary,

Project 2. *Apply a suitable approximate functional equation and Poisson summation to the expression obtained for the regular elliptic terms. Analyze the dominant term in this Poisson summation and look for cancellation with the contribution of the trivial representation.*

Remark 3.1. While most of the extant literature on the trace formula for groups larger than $GL(2)$ are presented in the language of the adèles—which is most natural—for the computations we have in mind we will quickly return to working over $\mathbf{R}$, which will be effected by the choice of test function as mentioned above.

3.1.3. *Non-elliptic terms.* As mentioned above, the expectation is that the limit of the expression (1) is zero. Indeed, you could say that this is a highly nontrivial exercise in calculating zero! Thus, all the other terms in (1)—the ones not associated to regular semisimple classes—must either vanish or cancel with the elliptic contribution. For GL(2) this is analyzed in §2.2–4 of [8] using the explicit expressions obtained in [7]

In our case, we would depend on having explicit expressions for the remaining terms following 3.1.1

Project 3. *Show that the contribution from elements which are not regular elliptic, and the continuous spectrum either vanish in the limit or cancel with the elliptic contribution.*

As we can see, a complete analysis requires having achieved the previous projects, but even partial progress, e.g., without knowing the elliptic terms, would already be valuable for understanding general case. Indeed, one of the goals of this project is to gain intuition for how the general case might look like, and what are the complications that must be surmounted.

3.2. **Sym^2 and $\otimes$-representations for GL(2).** The problem outlined in the previous setting gives the simplest example of the beyond endoscopy method for a higher rank group, another direction to pursue is more general representations r in the smallest rank, that is, $GL(2)$. The beyond endoscopy method for the symmetric square representation Sym^2 for $GL(2)$ was carried out in the thesis of A. Venkatesh [11] using the Kuznetsov trace formula.

The technical advantages of the Kuznetsov formula are twofold: the special representations, i.e., the so-called non-tempered representations, encountered in [1] as the trivial representation does do not occur in the Kuznetsov formula, hence need not be removed to get the formula we want. Additionally, the Kuznetsov formula is weighted by special values of the adjoint L-function $1/L(1,\pi,\mathrm{Ad})$—the adjoint representation is the symmetric square twisted by a determinant—conveniently cancels out the singularities arising from $L(1,\pi,\mathrm{Sym}^2)$ which our limiting expression is essentially evaluating. It is noted, however, that this 'miracle' is convenient but not necessary.

We would like to execute this method using the trace formula of Selberg rather than Kuznetsov:

Problem 2. *Carry out the method for $GL(2)$ and the symmetric square representation.*

Now, the reader should be aware that certain analytic difficulties encountered here have been pointed out both by Langlands [8] and Sarnak [10]. Instead of addressing this case directly, it may be possible to make headway here through the tensor product representation $\otimes$, which decomposes into the symmetric square Sym^2 and exterior square Λ^2. The tensor product L-function, or the Rankin-Selberg L-function, has been well studied

$$L(s,\pi_1,\pi_2,\otimes)=L(s,\pi\times\pi):=\sum_{n=1}^{\infty}\frac{a_{\pi_1}(n)b_{\pi_2}(n)}{n^s}$$

In particular, beyond endoscopy for the Rankin-Selberg L-functions of forms on $GL(2)$ has been studied, though incomplete, by P.E. Hermann [5] again through the Kuznetsov trace formula. In this setting one expect $L(s,\pi_1\times\pi_2)$ to have a pole at $s=1$ if and only if $\pi_2\simeq\tilde{\pi}_1$.

Remark 3.2. This result was previously proved using the converse theorem method on $GL(3)$ by Gelbart and Jacquet in 1978, followed by Sym^3 by Kim and Shahidi in 2002 and also Sym^4 by Kim.

Here, we would like to analyze the limit

$$\sum_{\pi} m_\pi(r)\mathrm{tr}(\pi(f))=\lim_{X\to\infty}\frac{1}{X}\sum_{n<X}\sum_{\pi}a_{\pi_1}(n)b_{\pi_2}(n)\mathrm{tr}(\pi(f)) \tag{3}$$

using the methods described above. The trace formula in this case would be that of $GL(2)$, as utilized in [8] and [1].

Project 4. *Analyze the limit (3).*

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