nite plane value as D and r_0 approach zero, and will become progressively larger than the infinite plane value as D and r_0 are increased. This trend is shown clearly in Fig. 6. Thus the experiment can be used either as an empirical attack on the problem for students who are not far enough along to go through the derivation, or as a satisfying confirmation of a less-than-obvious theoretical result.

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On the numbers of things and the distribution of first digits

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I answer the question "Why are there more small things in the world than large things?" in terms of a probabilisitc model of partitioning a conserved quantity. Benford's empirical rule for lists of numbers, that the proportion of numbers with first digit m is $\log_{10}(m+1) - \log_{10} m$, is an exact consequence of the model.

Why are there more small things in the world than large? Why, for instance, are there more pebbles in a rock slide than boulders, more raindrops in the atmosphere than lakes on the ground, and more provinces in the world than states? A general answer to these questions is because a few large things can be broken into many small things (e.g., boulders and pebbles), many small things can be condensed into a few large things (e.g., raindrops and lakes), and large things are composed of many small things (e.g., states and provinces). These observations may appear obvious. Yet, I believe, they are essentially the explanation of a somewhat mysterious phenomena referred to as "the peculiar distribution of first digits."[1]

It is a simple matter to illustrate the phenomena by inspecting a ranked list of the population of cities and towns in the United States found in a geographical atlas or almanac. In any one decade, say from one hundred thousand to one million, roughly a third of the numbers begin with one, a somewhat smaller number begin with two, an even smaller number begin with three and so on through the digits to nine. More surprisingly, the same pattern is observed in many disparate collections of numbers. Among those identified by the physicist Frank Benford in 1938 are the surface area of largest 335 rivers in the world and the street addresses of the first 342 persons listed in *American Men of Science*.[2] Counting the frequency with which each of the numbers 1–9 appear as first digits in some 20 000 numbers of this kind, he found evidence for the following rule. The first digit m appears with a probability $\log_{10}(m+1) - \log_{10} m$. Thus, according to Benford's rule the probability that an entry will start with 1, 2, 3, 4, 5, 6, 7, 8, or 9 is, respectively, 0.301, 0.176, 0.125, 0.097, 0.079, 0.067, 0.058, 0.051, or 0.046.

Physicists usually leave questions such as those posed in the opening paragraph to other specialists. Even so, these questions remind us of one we often do ask: What is the most probable way a conserved quantity can be partitioned into pieces subject to one or more other constraints? For instance, in determining the most probable velocity distribution of a gas one counts the number of *a priori* equally probable states, having the same total energy and number of atoms, which results in a particular distribution. The counting is done in accordance with a rule for distinguishing among states and the most probable distribution is the one corresponding to the largest number of states.

The model presented here is different. I claim that what we observe in the number distribution of the sizes of things, be they rocks in a pile of rubble or lakes, ponds, and droplets on the Earth's surface, or cities and towns, is not a most probable distribution but a sum or equivalently an average or expected distribution of pieces of a conserved quantity. In the derivation which follows I constrain the average only by specifying the largest and smallest possible piece of the conserved quantity. Also each distinct distribution is assumed *a priori* equally probable. The average distribution determined in this way turns out to have Benford's empirical rule as an exact consequence.

Consider first a conserved quantity of magnitude X which is broken into $n_j \Delta x_j$ pieces with magnitudes between x_j and $x_j + \Delta x_j$ for j taking on integer values between 1 and N. Here N is the number of divisions made in a closed domain defined by lower, x_l, and upper, x_u, bounds. Thus the set of numbers n_j, x_j, and Δx_j for all j defines a discrete distribution of fragments. Also,

$$X = \sum_{i=1}^{N} x_j n_j \Delta x_j, \tag{1}$$

expresses a conservation relation.

Now, consider all such distributions on the domain $x_l \leqslant x_j \leqslant x_u$ consistent with Eq. (1). Among possible distributions we include those with real n_j. Restricting n_j to the integers actually makes a general solution to this problem more difficult. The quantity X is the same for all these distributions but the number of fragments $\Sigma_{i=1}^{n} n_j \Delta x_j$ need not be. Our goal is to find the average or expected number density, $\bar{n}_j$, of each of the supposedly random numbers n_j subject to the conservation relation (1), the condition that $x_l \leqslant x_j \leqslant x_u$ for all j, and the ansatz that each distinct distribution is equally probable. Fortunately, we need not perform the averages explicitly, but can make use of the following simple theorems governing averages of constants (a,s) and random variables (x_1, etc.):

$$A: \quad \overline{ax_1} = a\bar{x}_1$$

$$B: \quad \overline{x_1 + x_2} = \bar{x}_1 + \bar{x}_2,$$

$$C: \quad \overline{x_1 + x_2 + \cdots x_n} = s$$

$$\rightarrow \bar{x}_1 = \bar{x}_2 = \cdots \bar{x}_n = s/n.$$

Theorem C follows from B and the intuition that since there is nothing to distinguish each term in the sum from any other term the average of each term must be equal.[3]

Finally, take the average of both sides of Eq. (1) and make use of theorems A, B, and C. Then it follows that

$$\bar{n}_1 x_1 \Delta x_1 = \bar{n}_2 x_2 \Delta x_2 = \cdots = X/N. \tag{2}$$

Therefore, the average distribution for discrete n_j and x_j is

$$\bar{n}_j = X/(N x_j \Delta x_j) \tag{3}$$

for all j. The limit of continuous x and $\bar{n}$ and vanishing Δx_j is most easily found when the distribution's domain is divided uniformly so that $\Delta x_1 = \Delta x_2 = \cdots \Delta x_N = (x_u - x_l)/N$. Then the limit $\Delta x_j \rightarrow 0$ of Eq. (3) with $x_u - x_l$ held constant results in $\bar{n}$ proportional to $1/x$ for $x_u \leqslant x \leqslant x_l$. I conjecture, without proof, that the division may also be made nonuniformly with essentially the same results, namely, a function $\bar{n}$ proportional to $1/x$ almost everywhere and integrable in the sense of Lebesgue.

It should be clear that a table which lists values of a quantity x which appear with a frequency proportional to $1/x$ will, when considered over an integral number of decades, obey Benford's rule. Specifically, the proportion of first digits m for entries between 10^p and 10^{p+1} will be

$$\int_m^{m+1} \frac{dx}{x} \Big/ \int_{10^p}^{10^{p+1}} \frac{dx}{x} = \ln\left(\frac{m+1}{m}\right) \Big/ \ln 10$$

$$= \log_{10}\left(\frac{m+1}{m}\right), \tag{4}$$

which is independent of the decade index p. This result immediately generalizes to numbers expressed in any base. Because Benford's rule is a consequence of the $1/x$ distribution, which in turn is a consequence of a conservation law and the present probabilistic model, I suspect that many tables obeying the rule are identical to or closely correlated with lists of pieces of conserved quantities. Alternatively, those lists that do not obey Benford's rule are probably not lists of pieces of a conserved quantity or violate the model in some other way.

Sometimes the identity of the conserved quantity relevant to a particular list is not obvious. An example is the frequency of first digits of a random selection of street addresses. If streets are treated as equal width strips of various lengths, then the sum of their areas is the quantity that is conserved upon partitioning. According to the present model their lengths x will be distributed as $1/x$. Consequently, the actual frequency of street addresses numbers although not necessarily distributed as $1/x$ will be weighted toward the smaller numbers.

Other explanations of Benford's rule exist. Furry and Hurwitz show that to good accuracy many distributions have Benford's rule as a consequence,[4] while Goudsmit and Furry offer the nonsequitur that "It [Benford's rule] is merely the result of our way of writing numbers..."[5] While the former is conceded the latter statement cannot be true since it fails to allow for distributions that do not obey Benford's rule. Other papers that argue the rule is a property of the number system itself suffer the same defect.[6]

More closely related to our own explanation is that of Roger Pinkham.[7] Pinkham first postulates the existence of some rule, not necessarily Benford's, which determines the probability of appearance of the first digits, then shows that if this rule is invariant to changes of scale (e.g., changing length measurements from feet to meters) it must be Benford's rule. The present explanation is consistent with Pinkham's theorem on the level of the distribution $f(x)$ producing the rule. Specifically, the amount of conserved quantity, $f(x)dx$, contained within an arbitrarily small range of sizes, dx, is scale invariant as long as $f(x) \propto 1/x$ for the sizes considered. Scale enters only with the distribution endpoints, that is with the smallest and largest possible pieces of the conserved quantity.

We can easily think of lists of numbers that are not lists of pieces of a conserved quantity: the ages of a random selection of people, the masses of elementary particles, as well as the square roots of the first n integers. Nevertheless, fragments of the conserved quantities mass, energy, volume, surface area, as well as others, are ubiquitous and Benford's law should apply to these on average.

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