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ABSTRACT.

CONTENTS

1. INTRODUCTION

2. SPECIAL CASE: $n = 1$ CASE

3. CONVERGENCE TO THE TRIVIAL FIXED POINT WHEN $b \leq \frac{(1-a)}{\sqrt{n}}$

We show in this subsection that if $b \leq (1-a)/\sqrt{n}$, then there is only one valid fixed point, the trivial fixed point.

Lemma 3.1. *Let $a, b \in (0,1)$ with $b < (1-a)/\sqrt{n}$, and let $\lambda_1 \geq \lambda_2$ denote the eigenvalues of the matrix $\begin{pmatrix} a\alpha & nb\beta \\ b\gamma & a\delta \end{pmatrix}$, where $\alpha, \beta, \gamma, \delta \in [0,1]$. Then $-1 < \lambda_1, \lambda_2 < 1$.*

Proof. The sum of the eigenvalues is the trace of the matrix (which is $a(\alpha + \delta)$), and the product of the eigenvalues is the determinant (which is $a^2\alpha\delta - nb^2\beta\gamma$). Thus the eigenvalues satisfy the characteristic equation

$$\lambda^2 - a(\alpha + \delta)\lambda + (a^2\alpha\delta - nb^2\beta\gamma). \tag{3.1}$$

The eigenvalues are therefore

$$\frac{a(\alpha+\delta) \pm \sqrt{a^2(\alpha+\delta)^2 - 4(a^2\alpha\delta - nb^2\beta\gamma)}}{2} = \frac{a(\alpha+\delta) \pm \sqrt{a^2(\alpha-\delta)^2 + 4nb^2\beta\gamma}}{2}. \tag{3.2}$$

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As the discriminant is positive, the eigenvalues are real. Since $a(\alpha+\delta) \geq 0$, we have $|\lambda_2| \leq \lambda_1$, where

$$0 \ \leq \ \lambda_1 \ = \ \frac{a(\alpha+\delta)+\sqrt{a^2(\alpha-\delta)^2+4nb^2\beta\gamma}}{2}. \tag{3.3}$$

As $\beta\gamma \leq 1$, $nb^2 < (1-a)^2$ and $\sqrt{u+v} \leq \sqrt{u}+\sqrt{v}$ for $u, v \geq 0$ we find

$$\begin{aligned} \lambda_1 \ &< \ \frac{a(\alpha+\delta)+\sqrt{a^2(\alpha-\delta)^2}+\sqrt{4(1-a)^2}}{2} \\ &= \ \frac{a(\alpha+\delta)+a|\alpha-\delta|+2(1-a)}{2} \\ &= \ \frac{2a\max(\alpha,\delta)+2(1-a)}{2} \\ &= \ 1-(1-\max(\alpha,\delta))\,a \ \leq \ 1, \end{aligned} \tag{3.4}$$

where the last claim follows from $a, \alpha, \delta \in [0,1]$. □

Theorem 3.2. *Assume $b < (1-a)/\sqrt{n}$. Then there is only one valid fixed point, the trivial fixed point (which may occur with multiplicity greater than 1). Further, iterates of any point converge to the trivial fixed point.*

Proof. We shall prove this by using the Mean Value Theorem and an eigenvalue analysis of the resulting matrix.

We have

$$f\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) \ = \ \begin{pmatrix} 1-(1-au)(1-bv)^n \\ 1-(1-av)(1-bu) \end{pmatrix}. \tag{3.5}$$

Let

$$c(t) \ = \ (1-t)\begin{pmatrix} 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} x \\ y \end{pmatrix}, \quad c'(t) \ = \ \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.6}$$

Thus $c(t)$ is the line connecting the trivial fixed point to $\begin{pmatrix} x \\ y \end{pmatrix}$, with $c(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $c(1) = \begin{pmatrix} x \\ y \end{pmatrix}$. Let

$$\mathcal{F}(t) \ = \ f(c(t)) \ = \ \begin{pmatrix} 1-(1-atx)(1-bty)^n \\ 1-(1-aty)(1-btx) \end{pmatrix}. \tag{3.7}$$

Then simple algebra (or the chain rule) yields

$$\mathcal{F}'(t) \ = \ \begin{pmatrix} a(1-bty)^n & nb(1-atx)(1-bty)^{n-1} \\ b(1-aty) & a(1-btx) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.8}$$

We now apply the one-dimensional chain rule twice, once to the x-coordinate function and once to the y-coordinate function. We find there are values t_1 and t_2 such that

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \ = \ \begin{pmatrix} a(1-bt_1y)^n & nb(1-at_1x)(1-bt_1y)^{n-1} \\ b(1-at_2y) & a(1-bt_2x) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.9}$$

To see this, look at the x-coordinate of $\mathcal{F}(t)$: $h(t) = 1-(1-atx)(1-bty)^n$. We have $h(1)-h(0)$ $= h(1) = h'(t_1)(1-0)$ for some t_1. As

$$\begin{aligned} h'(t_1) \ &= \ ax(1-bt_1y)^n + nby(1-at_1x)(1-bt_1y)^{n-1} \\ &= \ \left(a(1-bt_1y)^n,\ nb(1-at_1x)(1-bt_1y)^{n-1}\right)\cdot\begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned} \tag{3.10}$$

the claim follows; a similar argument yields the claim for the y-coordinate (though we might have to use a different value of t, and thus denote the value arising from applying the Mean Value Theorem here by t_2).

We therefore have

$$\begin{aligned} f\left(\left(\begin{array}{c} x \\ y \end{array}\right)\right) &= \left(\begin{array}{cc} a(1-bt_1y)^n & nb(1-at_1x)(1-bt_1y)^{n-1} \\ b(1-at_2y) & a(1-bt_2x) \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right) \\ &= A(a,b,x,y,t_1,t_2)\left(\begin{array}{c} x \\ y \end{array}\right). \end{aligned} \tag{3.11}$$

To show that f is a contraction mapping, it is enough to show that, for all a, b with $b < (1-a)/\sqrt{n}$ and all $x, y \in [0,1]$ that the eigenvalues of $A(a,b,x,y,t_1,t_2)$ are less than 1 in absolute value; however, this is exactly what Lemma 3.1 gives (note our assumptions imply that $\alpha = (1-bt_1y)^n$ through $\delta = (1-bt_2x)$ are all in $(0,1)$). Let us denote $\lambda_{\max}(a,b)$ the maximum value of λ_1 for fixed a and b as we vary $t_1, t_2, x, y \in [0,1]$. As we have a continuous function on a compact set, it attains its maximum and minimum. As λ_1 is always less than 1, so is the maximum. Here it is important that we allow ourselves to have $t_1, t_2 \in [0,1]$, so that we have a closed and bounded set; it is immaterial (from a compactness point of view) that $a, b \in (0,1)$ as they are fixed. As $0 < a, b < 1$, we have $\alpha, \beta, \gamma, \delta < 1$ and thus the inequalities claimed in Lemma 3.1 hold. For any matrix M we have $||Mv|| \le |\lambda_{\max}|||v||$; thus

$$\left|\left|f\left(\left(\begin{array}{c} x \\ y \end{array}\right)\right)\right|\right| \;\le\; \lambda_{\max}(a,b)\left|\left|\left(\begin{array}{c} x \\ y \end{array}\right)\right|\right|; \tag{3.12}$$

as $\lambda_{\max}(a,b) < 1$ we have a contraction map. Therefore any non-zero $\left(\begin{array}{c} x \\ y \end{array}\right)$ iterates to the trivial fixed point if $b < (1-a)/\sqrt{n}$ and $n \ge 2$. In particular, the trivial fixed point is the only fixed point (if not, $A(a,b,x,y,t_1,t_2)v = v$ for v a fixed point, but we know $||A(a,b,x,y,t_1,t_2)v|| < ||v||$ if v is not the zero vector). □

4. Convergence to a Unique Nontrivial Fixed Point when $b > \frac{(1-a)}{\sqrt{n}}$

Lemma 4.1. *Let $h_1, h_2 : [0,1] \to [0,1]$ be twice continuously differentiable functions such that $h_1(x)$ is convex up, $h_2(x)$ is concave up, $h_1(0) = h_2(0) = 0$ and $h_1(x) \neq h_2(x)$ for $x > 0$ sufficiently small. Then for at most two choices of x do we have $h_1(x) = h_2(x)$.*

Proof. The claim is trivial if there is only one point of intersection, so assume there are at least two. Without loss of generality we may assume $p > 0$ is the first point above zero where h_1 and h_2 agree. Such a smallest point exists by continuity, as we have assumed $h_1(x) \neq h_2(x)$ for $x > 0$ sufficiently small; if there are infinitely many points x_n where they are equal, let $p = \liminf_n x_n > 0$. (We technically do not need to prove this – we could take any two points where the functions agree and show there cannot be a third point larger than the first two where the functions agree.)

Because $h_1(x)$ is convex up, $h_1'(x)$ is increasing. By the mean value theorem there is a point $c_1 \in (0,p)$ such that $h_1'(c_1) = (h_1(p) - h_1(0))/(p-0) = h_1(p)/p$. As h_1' is increasing, we have $h_1'(p) > h_1(c_1)$; further, $h_1'(x) > h_1(c_1)$ for all $x \ge p$. As $h_2(x)$ is concave up, $h_2'(x)$ is decreasing. Again by the mean value theorem there is a point $c_2 \in (0,p)$ such that $h_2'(c_2) = ((h_2(p) - h_2(0))/(p-0) = h_2(p)/p$. As h_2' is decreasing, we have $h_2'(p) < h_2'(c_2)$, and in fact $h_2'(x) < h_2'(c_2)$ for all $x \ge p$. But $h_1'(c_1) = h_2'(c_2)$ (since $h_1(p) = h_2(p)$), so $h_1'(x) > h_2'(x)$ for all $x \ge p$. Thus there cannot be another point of intersection after p. □

For $b > (1-a)/\sqrt{n}$, we show that there is a unique, nontrivial valid fixed point and that all non-trivial iterates converge to that nontrivial fixed point. The existence and uniqueness proof involves looking at the intersection of two curves, one where the x-coordinate is unchanged under applying f, and one where the y-cooridnate is unchanged after applying f. One of these curves is concave up, the other convex up. The proof is completed by the following lemma.

Theorem 4.2. *Assume $a, b \in (0,1)$, $b > (1-a)/\sqrt{n}$ and $n \geq 2$. Then there exists a unique non-trivial, valid fixed point.*

Proof. We prove this through repeated applications of the Intermediate Value Theorem and continuity. Let

$$g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix} = f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) - \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.1}$$

Note $\begin{pmatrix} x \\ y \end{pmatrix}$ is a fixed point if and only if $g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = 0$.

We first look for partial fixed points, namely points where either the x or the y-coordinate is unchanged. These correspond to finding $\begin{pmatrix} x \\ y \end{pmatrix}$ with $g_1(x,y) = 0$ or $g_2(x,y) = 0$. We first analyze the set of pairs $(x,y) \in [0,1]^2$ where $g_1(x,y) = 0$. We have

$$g_1(x,y) = (1 - (1-ax)(1-by)^n) - x. \tag{4.2}$$

We immediately see that $g_1(0,0) = 0$, $g_1(0,y) > 0$ for $y \in (0,1]$, and $g_1(1,y) < 0$ for $y \in [0,1]$. Thus by the Intermediate Value Theorem, for each $y \in (0,1]$ there is a $\phi_1(y)$ such that $g_1(\phi_1(y), y) = 0$ and $\phi_1(y) \in [0,1]$. It is easy to see that $\phi_1(y)$ is a continuous function of y; in fact,

$$\begin{aligned} \phi_1(y) &= \frac{1-(1-by)^n}{1-a(1-by)^n} \\ \phi_1'(y) &= \frac{nb(1-a)(1-by)^{n-1}}{(1-a(1-by)^n)^2}. \end{aligned} \tag{4.3}$$

Note $\phi_1(y) \in [0,1]$: it is clearly positive, and $\frac{1-c}{1-ac} > 1$ for $c > 0$ only when $a > 1$. As $a, b \in (0,1)$, $\phi_1'(y) > 0$. Thus $\phi_1(y)$ is strictly increasing, and $\phi_1(0) = 0$. Further, we have for small y that $\phi_1(y) \approx \frac{nb}{1-a} y$. To see this, we note $(1-by)^n = 1 - nby + O(y^2)$ and substitute into (4.3). To aid in the analysis below, it is more convenient to re-write this as $y \approx \frac{1-a}{nb} x$ (as $\phi_1'(y) > 0$ we may use the inverse function theorem to write y as a function of x).

We analyze $g_2(x,y) = 0$ similarly. We find

$$g_2(x,y) = (1-(1-ay)(1-bx)) - y = 0. \tag{4.4}$$

Note $g_2(0,0) = 0$, $g_2(x,0) > 0$ for $x \in (0,1]$, and $g_2(x,1) < 0$ for $x \in [0,1]$. Solving yields

$$y = \phi_2(x) = \frac{bx}{1-a+abx}. \tag{4.5}$$

This is clearly continuously differentiable, and

$$\phi_2'(x) = \frac{b(1-a)}{(1-a+abx)^2} > 0. \tag{4.6}$$

Thus $\phi_2(x)$ is an increasing function of x. Further, for small x we have $y \approx \frac{b}{1-a} x$.

We now use the assumption that $b > (1-a)/\sqrt{n}$. Near the origin, $\phi_1(y)$ looks like the line $y = \frac{1-a}{nb} x$, while near the origin $\phi_2(x)$ looks like the line $y = \frac{b}{1-a} x$. If $\frac{1-a}{nb} < \frac{b}{1-a}$ then $\phi_2(x)$ is above $\phi_1(y)$ near the origin. Cross multiplying shows that this condition is equivalent to $b^2 > (1-a)/n$, or $b > (1-a)/\sqrt{n}$. Thus, for $a, b \in (0,1)$ and $b > (1-a)/\sqrt{n}$, the two curves $x = \phi_1(y)$ and $y = \phi_2(x)$ have at least two intersections in $[0,1]^2$; one is the trivial fixed point while the other is a non-trivial, valid fixed point. The existence of the second point of intersection follows from the intermediate value theorem (near the origin $y = \phi_2(x)$ is above $x = \phi_1(y)$; however, as $x \to 1$ we have $\phi_2(x)$ tends to a number strictly less than 1. Thus the curve $y = \phi_2(x)$ hits the line $x = 1$ below $(1,1)$. Similarly the curve $x = \phi_1(y)$ hits the line $y = 1$ to the left of $(1,1)$. Thus the two curves flip as to which is above the other, implying that there must be one point where the two curves are equal. This point is clearly a fixed point.

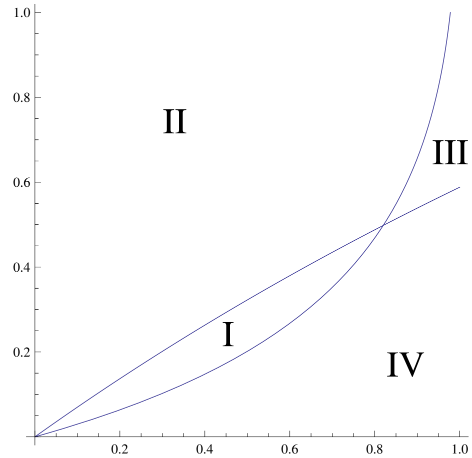

FIGURE 1. Four regions determined by ϕ_1 and ϕ_2.

We now show there are only two intersections (i.e., there is a unique, non-trivial valid fixed point). The proof follows from showing that $y = \phi_2(x)$ is concave up (concave increasing) and $x = \phi_1(y)$ is convex up (convex increasing). There are already two points of intersection, and by Lemma 4.1 there can be at most two points of intersection. Straightforward differentiation and some algebra gives

$$\begin{aligned} \phi_2''(x) &= \frac{-2ab^2(1-a)}{(1-a+abx)^2} < 0 \\ \phi_1''(y) &= -\frac{b^2n(1-a)(1-by)^{n-2}\cdot(n-1+a(1-by)^n+a(n+1)(1-by)^n)}{(1-a(1-by)^n)^3} < 0. \end{aligned} \tag{4.7}$$

Thus $y = \phi_2''(x)$ is concave up (since the second derivative is always negative and the first derivative is always positive: compare this to the standard parabola $y = -x^2$ when $x < 0$). As a function of y, $x = \phi_1(y)$ is also concave up (since its first derivative is positive and its second derivative is negative); however, we are interested in $y = \phi_1^{-1}(x)$ (the inverse function exists because the first derivative is positive). If $\phi_1(y)$ is concave up as a function of y then $\phi_1^{-1}(x)$ is convex up as a function of x. This follows because we are basically reflecting about the $x = y$ line, and this switches us from concave to convex (the function is obviously still increasing). The claim now follows from Lemma 4.1. □

Proof of convergence to this nontrivial fixed point relies on an analysis of the behavior of points in the regions defined by $\phi_1(x)$ and $\phi_2(y)$ above. See Figure 1 for

Lemma 4.3. *Points in region I strictly increase in x and y on iteration, and points in region III strictly decrease in x and y on iteration.*

Proof. A point $\begin{pmatrix} x \\ y \end{pmatrix}$ in region I satisfies the inequalities:

$$x < \frac{1-(1-by)^n}{1-a(1-by)^n} \tag{4.8}$$

and

$$y < \frac{bx}{1-a+abx} \tag{4.9}$$

By multiplying by the denominator on both sides for both inequalities, we find that:

$$x - ax(1-by)^n < 1-(1-by)^n \tag{4.10}$$

and

$$y - ay + abxy \ < \ bx. \tag{4.11}$$

Rearranging these terms gives:

$$x \ < \ 1 - (1-by)^n + ax(1-by)^n = 1 - (1-ax)(1-by)^n = f_1(x,y) \tag{4.12}$$

and

$$y \ < \ ay + bx - abxy = 1 - (1-ay)(1-bx) = f_2(x,y). \tag{4.13}$$

Thus, the x and y coordinates of the iterate of a point in region I are strictly greater than the x and y coordinates of the initial point.

The proof for points in region III is exactly analogous except with the inequalities fixed, so that:

$$x \ > \ \frac{1-(1-by)^n}{1-a(1-by)^n} \tag{4.14}$$

and

$$y \ > \ \frac{bx}{1-a+abx} \tag{4.15}$$

implies

$$x \ > \ 1 - (1-ax)(1-by)^n = f_1(x) \tag{4.16}$$

and

$$y \ > \ 1 - (1-ay)(1-bx) = f_2(y), \tag{4.17}$$

i.e. the x and y coordinates of the iterate of a point in region III are strictly less than the x and y coordinates of the initial point. □

Lemma 4.4. *Points in region I iterate inside region I, and points in region III iterate inside region III.*

Proof. We prove that for a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in region I, its iterated x-coordinate satisfies 4.14 and its iterated y-coordinate satisfies 4.15.

X**-Coordinate Iteration:** We must show that:

$$1 - (1-ax)(1-by)^n \ < \ \frac{1-(1-b(1-(1-ay)(1-bx)))^n}{1-a(1-b(1-(1-ay)(1-bx)))^n}. \tag{4.18}$$

Since $\begin{pmatrix} x \\ y \end{pmatrix}$ is in region I, we know that

$$x \ < \ 1 - (1-ax)(1-by)^n, \tag{4.19}$$

which implies that

$$\frac{x}{1-(1-ax)(1-by)^n} \ < \ 1. \tag{4.20}$$

Since $0 < a, b, y < 1$, we know that $a(1-by)^n > 0$. Thus,

$$1 - \frac{ax(1-by)^n}{1-(1-ax)(1-by)^n} \ > \ 1 - a(1-by)^n. \tag{4.21}$$

We simplify the left side of the inequality:

$$\begin{aligned}
\frac{1-(1-ax)(1-by)^n}{1-(1-ax)(1-by)^n} - \frac{ax(1-by)^n}{1-(1-ax)(1-by)^n} \quad &> \quad 1 - a(1-by)^n \\
\frac{1-(1-by)^n + ax(1-by)^n}{1-(1-ax)(1-by)^n} - \frac{ax(1-by)^n}{1-(1-ax)(1-by)^n} \quad &> \\
\frac{1-(1-by)^n}{1-(1-ax)(1-by)^n} \quad &>
\end{aligned} \tag{4.22}$$

Finally, we rearrange the inequality:

$$\frac{1-(1-by)^n}{1-a(1-by)^n} > 1-(1-ax)(1-by)^n \tag{4.23}$$

For the second part of the proof, recall that

$$y < 1-(1-ay)(1-bx) \tag{4.24}$$

which implies

$$(1-b(1-(1-ay)(1-bx)))^n < (1-by)^n \tag{4.25}$$

Now we let $(1-b(1-(1-ay)(1-bx)))^n = c$ and $(1-by)^n = c+\delta$ where $0 < c < 1$ and $\delta > 0$ such that $c < c+\delta < 1$. Then we can write

$$\begin{aligned} -\delta &< -a\delta \\ 1-c-\delta-ac+ac^2+ac\delta &< 1-c-ac+ac^2-a\delta+a\delta c \\ (1-ac)(1-c-\delta) &< (1-ac-a\delta)(1-c) \\ \frac{1-(c+\delta)}{1-a(c+\delta)} &< \frac{1-c}{1-ac}. \end{aligned} \tag{4.26}$$

Thus

$$\frac{1-(1-b(1-(1-ay)(1-bx)))^n}{1-a(1-b(1-(1-ay)(1-bx)))^n} > \frac{1-(1-by)^n}{1-a(1-by)^n}. \tag{4.27}$$

The desired result follows from 4.23, 4.27, and transitivity.

Y**-Coordinate Iteration:** We must show that:

$$1-(1-ay)(1-bx) < \frac{b(1-(1-ax)(1-by)^n)}{1-a+ab(1-(1-ax)(1-by)^n)} \tag{4.28}$$

Since $\begin{pmatrix} x \\ y \end{pmatrix}$ is in region I, we know that:

$$x < 1-(1-ax)(1-by)^n \tag{4.29}$$

$$y < 1-(1-ay)(1-bx) \tag{4.30}$$

which implies that

$$\frac{y}{1-(1-ay)(1-bx)} < 1 \tag{4.31}$$

Now since $0 < a, b, x < 1$, we know that $abx - a < 0$. Thus,

$$1+\frac{y(abx-a)}{1-(1-ay)(1-bx)} > 1-a+abx \tag{4.32}$$

We simplify the left side of the inequality:

$$\begin{aligned}\frac{1-(1-ay)(1-bx)}{1-(1-ay)(1-bx)}+\frac{y(abx-a)}{1-(1-ay)(1-bx)} &> 1-a+abx\\ \frac{ay+bx-abxy}{1-(1-ay)(1-bx)}+\frac{abxy-ay}{1-(1-ay)(1-bx)} &>\\ \frac{bx}{1-(1-ay)(1-bx)} &> \end{aligned}\tag{4.33}$$

Finally, we rearrange the inequality:

$$\frac{bx}{1-a+abx} > 1-(1-ay)(1-bx)\tag{4.34}$$

For the second part of the proof, recall that

$$x < 1-(1-ax)(1-by)^n\tag{4.35}$$

This allows us to write $1-(1-ax)(1-by)^n = x+c$ for some $c>0$ such that $x < x+c < 1$. Since $c>0$ and $a,b<1$ we see that

$$\begin{aligned} bc-abc &> 0\\ bx+bc-abx-abc+ab^2x^2+ab^2xc &> bx-abx+ab^2x^2+ab^2xc\\ b(x+c)(1-a+abx) &> bx(1-a+ab(x+c))\end{aligned}\tag{4.36}$$

Thus

$$\frac{b(x+c)}{1-a+ab(x+c)} > \frac{bx}{1-a+abx}\tag{4.37}$$

That is,

$$\frac{b(1-(1-ax)(1-by)^n)}{1-a+ab(1-(1-ax)(1-by)^n)} > \frac{bx}{1-a+abx}\tag{4.38}$$

The desired result follows from 4.34, 4.38, and transitivity. □

Lemma 4.5. *All nontrivial points in regions I and III converge to the nontrivial fixed point.*

Proof. Consider any nontrivial point $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in region I. Define a sequence $z_{t+1} = f(z_t)$. By 4.3, we know that z_t is monotonically increasing. Furthermore, we know that z_t is bounded by $\begin{pmatrix} x_f \\ y_f \end{pmatrix}$. Thus, z_t must converge. Suppose it converges to z', i.e. $\lim_{t\to\infty} z_t = z'$. We consider the iterate of z'.

$$f(z') = f\left(\lim_{t\to\infty} z_t\right) = \lim_{t\to\infty} f(x_t) = \lim_{t\to\infty} z_{t+1} = \lim_{t\to\infty} z_t = z'.\tag{4.39}$$

Thus, z' is a fixed point. Since $z_0 > \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and z_t is increasing, z' cannot be the trivial fixed point. Thus z' must be the unique nontrivial fixed point. For region III, we have a monotonically decreasing and bounded sequence z_t that must thus converge to a fixed point. By 4.4, this fixed point must be in region III and thus can only be the unique nontrivial fixed point. □

Theorem 4.6. *Any nontrivial point in $[0,1]\times[0,1]$ converges to the unique nontrivial fixed point.*

Proof. Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be a point in $(0,1] \times (0,1]$, and choose $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$ in Region I and $\begin{pmatrix} x_s \\ y_s \end{pmatrix}$ in Region III such that $x_i \leq x \leq x_s$ and $y_i \leq y \leq y_s$. Define the sequence $z_t = (z_t(x), z_t(y))$ such that $z_{t+1} = (z_{t+1}(x), z_{t+1}(y))$, with $z_{t+1}(x) = f_1(z_t(x), z_t(y))$ and $z_{t+1}(y) = f_2(z_t(x), z_t(y))$. Let $z_0(x) = x$ and $z_0(y) = y$. We show by induction that $z_t(x_i) \leq z_t(x) \leq z_t(x_s)$ and $z_t(y_i) \leq z_t(y) \leq z_t(y_s)$ for all $t \in \mathbb{N}$.

The base case is given by our choice of $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$ and $\begin{pmatrix} x_s \\ y_s \end{pmatrix}$, so we proceed to show the inductive step. Suppose that we have $z_t(x_i) \leq z_t(x)$ and $z_t(y_i) \leq z_t(y)$. Then

$$1 - az_t(x_i) \geq 1 - az_t(x) \tag{4.40}$$

and

$$1 - bz_t(y_i) \geq z_t(y) \tag{4.41}$$

which implies that

$$(1 - az_t(x_i))(1 - bz_t(y_i)^n \geq (1 - az_t(x))(1 - bz_t(y))^n \tag{4.42}$$

for any $n \geq 1$. Then

$$1 - (1 - az_t(x_i))(1 - bz_t(y_i)^n \leq 1 - (1 - az_t(x))(1 - bz_t(y))^n. \tag{4.43}$$

That is, $z_{t+1}(x_i) \leq z_{t+1}(x)$. Furthermore, we have that

$$1 - az_t(y_i) \geq 1 - az_t(y) \tag{4.44}$$

and

$$1 - bz_t(x_i) \geq 1 - bz_t(x) \tag{4.45}$$

which implies that

$$(1 - az_t(y_i))(1 - bz_t(x_i) \geq (1 - az_t(y))(1 - bz_t(x)). \tag{4.46}$$

Then

$$1 - (1 - az_t(y_i))(1 - bz_t(x_i) \leq 1 - (1 - az_t(y))(1 - bz_t(x)). \tag{4.47}$$

That is, $z_{t+1}(y_i) \leq z_{t+1}(y)$. By a similar argument, we see that $z_t(x) \leq z_t(x_s)$ and $z_t(y) \leq z_t(y_s)$ implies that $z_{t+1}(x) \leq z_{t+1}(x_s)$ and $z_{t+1}(y) \leq z_{t+1}(y_s)$.
Thus $z_t(x_i) \leq z_t(x) \leq z_t(x_s)$ and $z_t(y_i) \leq z_t(y) \leq z_t(y_s)$ for all $t \in \mathbb{N}$. Taking the limit, we have

$$\lim_{t\to\infty} z_t(x_i) \leq \lim_{t\to\infty} z_t(x) \leq \lim_{t\to\infty} z_t(x_s) \tag{4.48}$$

and

$$\lim_{t\to\infty} z_t(y_i) \leq \lim_{t\to\infty} z_t(y) \leq \lim_{t\to\infty} z_t(y_s) \tag{4.49}$$

Since $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$ is in Region I and $\begin{pmatrix} x_s \\ y_s \end{pmatrix}$ is in Region III, the inequalities become

$$x_f \leq \lim_{t\to\infty} z_t(x) \leq x_f \tag{4.50}$$

and

$$y_f \leq \lim_{t\to\infty} z_t(y) \leq y_f. \tag{4.51}$$

Thus $lim_{t\to\infty} z_t(x) = x_f$ and $lim_{t\to\infty} z_t(y) = y_f$, that is, $\begin{pmatrix} x \\ y \end{pmatrix}$ iterates to $\begin{pmatrix} x_f \\ y_f \end{pmatrix}$. □

5. Behavior

Corollary 5.1. *The amount of time it takes for all points to converge is the maximum of the time it takes* $\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *to converge, for* $\epsilon_1, \epsilon_2 \to 0$.

Conjecture 5.2. *Points in region II and IV exhibit one of two behaviors, dependent on* a, b, n. *Either:*

(1) *All points in region II iterate outside region II and all points in region IV iterate outside region IV ("flipping behavior"), or*
(2) *All points in region II iterate outside region IV and all points in region IV iterate outside region II ("non-flipping behavior")*

6. Generalized Star Graphs

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