

THOUGHTS ON SPECIAL MAPS RELATED TO MODELLING INFECTIONS

LEO KONTOROVICH, STEVEN J. MILLER, AND AMITABHA ROY

ABSTRACT. We investigate some consequences of a map that arises in investigations of models of spreads of infections.

1. INTRODUCTION AND NOTATION

In [KP] the following equation is shown to be related to the propagation of infections:

$$f_n \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} 1 - (1 - ax)(1 - by)^n \\ 1 - (1 - ay)(1 - bx) \end{pmatrix} \quad (1.1)$$

(where we have replaced d with $1 - a$). We study $f_n : [0, 1]^2 \rightarrow [0, 1]^2$.

When n is fixed, for notational convenience we often write f for f_n . We always have $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a fixed point; we shall call this the trivial fixed point, and any other fixed point is called non-trivial. A valid or admissible fixed point is one in $[0, 1]^2$.

In our arguments below we constantly use $0 < a, b < 1$. Some of the most important consequences are the positivity of certain expressions, as well as $\frac{1}{a}$ and $\frac{1}{b}$ are both greater than 1.

These are rough notes right now. First we give Steve's arguments which completely analyze the $n = 1$ case. We then give some arguments for $n = 2$. In particular, we give Amitabha's argument proving Steve's conjecture, and then some arguments of Steve describing the nature of the fixed points. Needless to say, these are very rough notes!

2. SPECIAL CASE: $n = 1$

We quickly sketch some of the arguments and results from the $n = 1$ case, as this suggests possible approaches to handle general n . In this case, [KP] shows that it suffices to consider a one-variable problem, namely $f(x) = 1 - (1 - ax)(1 - bx)$. This is because when $n = 1$ we cannot distinguish a spoke from the central node.

2.1. Fixed Points.

Lemma 2.1. *The fixed points of f are 0 and $\frac{a+b-1}{ab}$. If $a + b \leq 1$ there is only one fixed point in $[0, 1]$, namely 0. If $a + b > 1$ then there is a second fixed point in $(0, 1)$.*

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Proof. We have

$$\begin{aligned}
f(x) - x &= 1 - (1 - ax)(1 - bx) - x \\
&= -abx^2 + (a + b)x - x \\
&= x(abx - (a + b - 1)) \\
&= abx \left(x - \frac{a + b - 1}{ab} \right). \tag{2.1}
\end{aligned}$$

As the fixed points are when $f(x) - x = 0$, the first half of the lemma is clear.

We must show $\frac{a+b-1}{ab} \in (0, 1)$. Clearly we need $a+b > 1$; thus in this case $\frac{a+b-1}{ab} > 0$. To show it is at most 1 it suffices to show $a + b - 1 < ab$ or $a + b - 1 - ab < 0$. As $a < 1$ we have

$$\begin{aligned}
a + b - 1 - ab &= a - ab + b - 1 \\
&= a(1 - b) - (1 - b) \\
&= (a - 1)(1 - b) < 0. \tag{2.2}
\end{aligned}$$

□

Remark 2.2. *The above argument is common in these investigations. Namely, after some (moderately clever?) algebra we can easily determine the sign of the relevant quantities.*

2.2. Derivative. Recall $f(x) = 1 - (1 - ax)(1 - bx)$. Thus

Lemma 2.3. *If $a + b \leq 1$ then $|f'(x)| \leq 1/2$ for all x ; if $a + b > 1$ then $f'(x) > 0$ for all x .*

Proof. We have

$$\begin{aligned}
f'(x) &= a(1 - bx) + b(1 - ax) \\
&= (a + b) - 2abx \\
&= ab \left(\frac{a + b}{ab} - 2x \right). \tag{2.3}
\end{aligned}$$

Note the first derivative is decreasing with increasing x .

If $a + b \leq 1$ then

$$|f'(x)| = |a + b - 2abx| < |1/2 - (a + b)| \leq 1/2 \tag{2.4}$$

(note $a + b \leq 1$ implies $ab \leq 1/4$).

Assume now $a + b > 1$. When $x = 0$ we have $f'(0) = a + b > 1$. When $x = 1$ we have $f'(1) = a + b - 2ab$. Note

$$a + b - 2ab = a - ab + b - ab = a(1 - b) + b(1 - a) > 0. \tag{2.5}$$

Thus the first derivative is always positive. □

Remark 2.4. *A trivial argument now shows that if $a + b \leq 1$ then we have a contraction map, and everything converges to the trivial fixed point. Thus we shall always assume below that $a + b > 1$, ie that we have a non-trivial, valid fixed point.*

Lemma 2.5. *If $a + b > 1$ then we have $f'(1) < 1$.*

Proof. This follows immediately from

$$f'(1) = a(1 - b) + b(1 - a) < 1 - b + b = 1. \quad (2.6)$$

□

The reason it is important to note that $f'(1) < 1$ is that we want to show that f is a contraction map, at least for a subset of $[0, 1]$. Let x_f denote the fixed point $\frac{a+b-1}{ab}$. By the mean value theorem we have

$$f(x) - f(x_f) = f'(\xi)(x - x_f), \quad \xi \in [x_f, x]; \quad (2.7)$$

if $x < x_f$ then we should write $[x, x_f]$ for the interval. As $f(x_f) = x_f$, we can easily see what happens to a point x under f :

$$x \rightarrow f(x) = x_f + f'(\xi)(x - x_f). \quad (2.8)$$

Thus if x starts *above* x_f then $f(x)$ is above x_f (because the derivative is always positive and $x > x_f$); if x starts *below* x_f then $f(x)$ is below x_f (because the derivative is always positive and $x < x_f$).

This suggests that we should think of f as a contraction map; the problem is we need to show the existence of a $\delta \in (0, 1)$ such that $|f'(x)| \leq 1 - \delta$. If this were true, then by the Mean Value Theorem we would immediately have f is a contraction. Unfortunately, the derivative can be larger than 1; for example, when $x = 0$ we have $f'(0) = a + b > 1$. Thus for a small interval about $x = 0$ we do not have a contraction.

A little algebra determines where f is a contraction. We must find x_c such that $f'(x_c) = 1$; as f' is decreasing then the interval $[x_c + \epsilon, 1]$ will work for any $\epsilon > 0$. We have

$$1 = f'(x_c) = a + b - 2abx_c \quad (2.9)$$

implies

$$x_c = \frac{a + b - 1}{2ab} = \frac{x_f}{2}. \quad (2.10)$$

For more on contraction maps, see for example [Rud]. We summarize our results for later use:

Lemma 2.6. *Let $a + b > 1$. The first derivative is decreasing on $[0, 1]$; thus its maximum is $f'(0) = a + b > 1$ and its minimum is $f'(1) < 1$. Further, $f'(x) > 1$ for $x \in [0, x_c)$, $f'(x_c) = 1$ and $f'(x) < 1$ for $x \in (x_c, 1]$. Note $f'(x) > 0$.*

Proof. That $f'(x)$ is decreasing follows from (2.3); the claims on $f'(0)$ and $f'(1)$ are immediate from the other lemmas. The rest follows from our choice of x_c . □

2.3. Dynamical Behavior. Remember we define x_c so that $f'(x_c) = 1$. Further $f'(x)$ is monotonically decreasing.

Theorem 2.7. *Let $x_0 \in (0, 1]$ and assume $a + b > 1$. Let $x_{m+1} = f(x_m)$. Then $\lim_{m \rightarrow \infty} x_m = x_f$, where x_f is the non-trivial, valid fixed point.*

Proof. If $x = 0$ then all iterates stay at 0. For any $\epsilon > 0$, if $x \in [x_c + \epsilon, 1]$ then f is a contraction map, and the iterates of x converge to x_f , the unique non-zero fixed point. As this holds for all $\epsilon > 0$, we see that the iterates of any $x \in (x_c, 1]$ converge to x_f .

We are left with $x \in (0, x_c]$. As $f'(x)$ is always greater than 1 on $(0, x_c)$, if $x \in (0, x_c]$ then $f(x) > x$. The proof is straightforward. By the Mean Value Theorem we have

$$f(x) = f(0) + f'(\xi)x, \quad \xi \in (0, x_c). \quad (2.11)$$

It is very important that $\xi \in (0, x_c)$ and not in $[0, x_c]$. The reason is that $f'(x) > 1$ in $(0, x_c)$ but $f'(x_c) = 1$ (see Lemma 2.6). As $f(0) = 0$ we have for all $x \in (0, x_c]$ that

$$f(x) = 0 + f'(\xi)x > x. \quad (2.12)$$

If for some $x \in (0, x_c]$ an iterate is in $(x_c, 1]$ then by earlier arguments the future iterates converge to x_f .

Thus we are reduced to the case of an $x \in (0, x_c]$ such that all iterates stay in $(0, x_c]$. We claim this cannot happen. As this is a monotonically increasing, bounded sequence, it must converge. Specifically, fix an $x \in (0, x_c)$. Let $x_1 = f(x)$ and in general $x_{m+1} = f(x_m)$. Assume all $x_m \in (0, x_c)$ (if ever an $x_m = x_c$ then $x_{m+1} = f(x_c) > x_c = x_m$ and the claim is clear). Thus $\{x_m\}$ is a monotonically increasing bounded sequence, and hence (compactness or the Archimedean property) converges, say to $\tilde{x} < x_c$. As f is continuous, as x_m converges to \tilde{x} we must have $f(x_m)$ converges to $f(\tilde{x})$; in other words, $\lim_{m \rightarrow \infty} x_m = \tilde{x}$ implies $\lim_{m \rightarrow \infty} f(x_m) = f(\tilde{x})$. But $f(x_m) = x_{m+1}$; thus x_m and $f(x_m)$ have the same limit! By regarding the sequence as x_m we see the limit is \tilde{x} ; by regarding the sequence as $f(x_m)$ we see the limit is $f(\tilde{x})$. However, $\tilde{x} < x_c$, so by the Mean Value Theorem $f(\tilde{x}) > \tilde{x}$ (though all we need is that $f(\tilde{x}) \neq \tilde{x}$). As the sequence $\{x_m\}$ cannot converge to two distinct numbers, our assumption that $\{x_m\}$ converged to an $\tilde{x} \in (0, x_c)$ must be false, proving the claim. Thus, if $x \in (0, x_c)$, eventually an iterate of x is at least x_c . By our previous analysis, we know that future iterates converge to x_f . \square

Remark 2.8. *Note the above proof required us to be very careful. Specifically, we used the fact that $f'(x) > 1$ for $x \in (0, x_c]$ to show that such x are repelled from the fixed point 0, and then we used the fact that $f'(x) < 1$ for $x \in (x_c, 1]$ to show such points are attracted by the non-zero fixed point x_f . Arguments of this nature can be generalized.*

Remark 2.9. *We could also remark that the point \tilde{x} would have to be a fixed point, which is impossible. We chose the above proof as it works for one-dimensional generalizations of this problem without requiring knowledge of the locations of fixed points.*

3. NEXT CASE: $n = 2$: RESULTS FROM AMITABHA

Steve conjectured that the behavior is as follows (for general n): if $b < \frac{\sqrt{n}}{n}(1-a)$ then the only valid fixed point is the trivial one, if $b = \frac{\sqrt{n}}{n}(1-a)$ then the trivial fixed point is a fixed point with multiplicity at least two (for $n \geq 2$), and if $b > \frac{\sqrt{n}}{n}(1-a)$ then there is also a valid non-trivial fixed point. In this section we give Amitabha's algebraic analysis of the $n = 2$ case, proving the conjecture. In the next section we analyze the dynamical behavior.

Crucial in our analysis is the following lemma from Steve:

Lemma 3.1. *Fix $a, b \in (0, 1)$. Let $\begin{pmatrix} x \\ y \end{pmatrix}$ denote a fixed point of f . If $0 \leq x \leq 1$ then $0 \leq y \leq 1$. Thus in order to determine if a fixed point is valid, it suffices to check the x -coordinate (or show the y -coordinate is invalid).*

Proof. If $\begin{pmatrix} x \\ y \end{pmatrix}$ is a fixed point, then looking at the y -coordinate of $f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$ gives

$$1 - (1 - ay)(1 - bx) - y = 0 \quad (3.1)$$

Simple algebra yields

$$y = \frac{bx}{1 - a + abx}. \quad (3.2)$$

We first show the denominator is always positive. We have $abx \leq a$ because $0 \leq a, b, x \leq 1$. Therefore $1 - a + abx \in [1 - a, 1]$. As the numerator is clearly non-negative, we see $y > 0$.

We now prove $y \leq 1$. As $0 \leq bx \leq b$, we have

$$y = \frac{bx}{1 - a + abx} \leq \frac{bx}{(1 - a)bx + abx} = \frac{1}{1 - a + a} = 1, \quad (3.3)$$

which proves $y \leq 1$. \square

3.1. Notation. We want to find the range of values of a and b , where $a, b \in [0, 1]$ such that the map

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 - (1 - ax)(1 - by)^2 \\ 1 - (1 - ay)(1 - bx) \end{pmatrix}. \quad (3.4)$$

has a fixed point, i.e. $f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$ where we require $x, y \in [0, 1]$. To dispose of trivial cases, we require $a \neq 0$ and $b \neq 0$ in the definition above. This is equivalent to solving the simultaneous equations:

$$\begin{aligned} x &= 1 - (1 - ax)(1 - by)^2 \\ y &= 1 - (1 - ay)(1 - bx) \end{aligned} \quad (3.5)$$

There are three solutions to this equation system (see attached Mathematica file or solve the associated quadratic)

$$\left\{ (0, 0), \left(\frac{n_1 - \sqrt{n_2}}{n_3}, \frac{-m_1 + \sqrt{n_2}}{m_3} \right), \left(\frac{n_1 + \sqrt{n_2}}{n_3}, -\frac{m_1 + \sqrt{n_2}}{m_3} \right) \right\}$$

where

$$\begin{aligned} n_1 &= a^3 + b^3 - 2a^2(2 + b) + a(2 + 2b - 2b^2) \\ n_2 &= b^2(-4a^3(-1 + b) + b^4 - 4a(-1 + b)(1 + b)^2 + 8a^2(-1 + b^2)) \\ m_1 &= -2a^2b + b^3 + 2ab(1 + b) \\ m_3 &= 2a(-1 + a - b)b^2 \\ n_3 &= 2ab(a^2 + b^2 - a(1 + 2b)). \end{aligned} \quad (3.6)$$

Thus, in addition to the trivial fixed point, there are two other fixed points (which may or may not be valid, and which may or may not be non-trivial):

$$\begin{aligned} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= \begin{pmatrix} \frac{n_1 - \sqrt{n_2}}{n_3}, \frac{-m_1 + \sqrt{n_2}}{m_3} \end{pmatrix}^T \\ \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} \frac{n_1 + \sqrt{n_2}}{n_3}, -\frac{m_1 + \sqrt{n_2}}{m_3} \end{pmatrix}^T. \end{aligned} \quad (3.7)$$

We analyze

$$\begin{aligned} x_1 &= \frac{n_1 - \sqrt{n_2}}{n_3} \\ y_2 &= -\frac{m_1 + \sqrt{n_2}}{m_3}. \end{aligned} \quad (3.8)$$

We shall show that y_2 is never in $[0, 1]$, and thus we need not worry about the fixed point $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. We shall see that x_1 is sometimes valid, sometimes not.

Convention: When we write $\sqrt{\alpha}$ for $\alpha \in \mathbb{R}$, we always mean the positive square root. For example, if we have an inequality $\beta + \sqrt{\alpha} < 0$, this necessarily implies that $\beta < 0$.

The following identities are easily verified using mathematica (see the section marked Identities in attached notebook).

$$n_1^2 - n_2 = 4(-1 + a)((a - 1)^2 - 2b^2)((a - b)^2 - a) \quad (3.9)$$

$$(n_3 - n_1)^2 - n_2 = 4(-1 + a)a(-1 + b)^2(1 - a + b)^2((a - b)^2 - a) \quad (3.10)$$

$$n_1 - n_3 = b^3 + 2a(1 - b)(b - a + 1)^2 \quad (3.11)$$

$$m_1 + m_3 = b(b^2 + 2a(1 - b)(1 + b - a)) \quad (3.12)$$

We first need some technical lemmas (See the section marked Plots in attached notebook).

Lemma 3.2. *For all $a, b \in (0, 1)$, $n_2 > 0$.*

Proof. Since

$$n_2 = b^2(b^4 + 4a(1 - b)(a - 1 - b)^2)$$

and every term in the summand is positive, $n_2 > 0$. \square

Remark: Because of Lemma 3.2, we can refer to $\sqrt{n_2}$ in our formulas without concerns about definability. Similarly, we may assume that $n_3 \neq 0$ since otherwise both non-zero fixed points become undefined; i.e., $b \neq a \pm \sqrt{a}$. Note we never need to worry about $b = a - \sqrt{a}$, as the right hand side is negative because $a \in (0, 1)$. Thus the only potential problem points are when $b = a + \sqrt{a}$. If $a + \sqrt{a} \leq 1$ then $a \leq (3 - \sqrt{5})/2 \approx .38$.

Lemma 3.3. *For all $a, b \in (0, 1)$, we have $n_1 - n_3 > 0$.*

Proof. Immediate since every summand on the right side of Equation 3.11 is positive. \square

3.2. Analysis of x_1 .

Lemma 3.4. *If $b < -\frac{1}{\sqrt{2}}(a - 1)$, then $n_1 > 0$.*

Proof. We have

$$\begin{aligned} n_1 &= 2(-1 + a)^2 a + 2(1 - a)ab - 2ab^2 + b^3 \\ &\geq 4b^2 a + 2(\sqrt{2}b)ab - 2ab^2 + b^3 \\ &\quad (\text{using } (a - 1)^2 > 2b^2 \text{ and } (1 - a) > \sqrt{2}b) \\ &\geq b^3 + 2(1 + \sqrt{2})ab^2 \end{aligned}$$

Since each summand in the last expression is positive, we have $n_1 > 0$. \square

We say that a real number x is admissible if $0 < x < 1$, otherwise it is inadmissible. We prove the following theorem:

Theorem 3.5. *If $b < -\frac{1}{\sqrt{2}}(a - 1)$, then x_1 is inadmissible.*

Proof. The hypothesis implies that $2b^2 < (a - 1)^2$, since both b and $-\frac{1}{\sqrt{2}}(a - 1)$ are positive. We argue by cases:

$(n_3 > 0)$: Note that this implies that $(a - b)^2 > a$. Thus from Equation (3.9), we have $n_1^2 < n_2$, so that $-\sqrt{n_2} < n_1 < \sqrt{n_2}$. This implies that the numerator of x_1 , i.e. $n_1 - \sqrt{n_2}$ is negative, while the denominator, i.e., n_3 , is positive, so $x_1 < 0$ is inadmissible.

$(n_3 < 0)$: This implies that $(a - b)^2 < a$. Thus $n_1^2 > n_2$ and so either $n_1 > \sqrt{n_2}$ or $n_1 < -\sqrt{n_2}$. If $n_1 > \sqrt{n_2}$, then the numerator of x_1 is positive, while the denominator is negative, so $x_1 < 0$, which makes it inadmissible. If instead, $n_1 < -\sqrt{n_2}$, then $n_1 + \sqrt{n_2} < 0$. But this means that $n_1 < 0$ violating Lemma 3.4. \square

Theorem 3.6. *If $b > -\frac{1}{\sqrt{2}}(a - 1)$, then x_1 is admissible.*

Proof. The hypothesis implies that $2b^2 > (a - 1)^2$, since both b and $-\frac{1}{\sqrt{2}}(a - 1)$ are positive. We argue by cases:

$(n_3 > 0)$: Note that this implies that $(a - b)^2 > a$. Now Equation (3.10) implies that $(n_3 - n_1)^2 < n_2$. Since $n_1 - n_3 > 0$ (Lemma 3.3), this implies $n_1 - n_3 < \sqrt{n_2}$, so that $n_1 - \sqrt{n_2} < n_3$ and so $x_1 < 1$, as required. We now show that $x_1 > 0$: since $n_3 > 0$, it suffices to prove that $n_1 > \sqrt{n_2}$. Our hypothesis implies that $n_1^2 > n_2$ so either $n_1 > \sqrt{n_2}$ or $n_1 < -\sqrt{n_2}$. The latter option cannot arise because $n_1 > n_3$ and $n_3 > 0$, so $n_1 > 0$. Thus we have $n_1 > \sqrt{n_2}$ and so $x_1 > 0$, making x_1 admissible.

$(n_3 < 0)$: This implies that $(a - b)^2 < a$. Equation (3.10) implies that $(n_3 - n_1)^2 > n_2$. Since $n_1 - n_3 > 0$ (Lemma 3.3), we must have $n_1 - n_3 > \sqrt{n_2}$ and so $n_1 - \sqrt{n_2} > n_3$. Since $n_3 < 0$, when we divide by n_3 the sign reverses so that

$$x_1 = \frac{n_1 - \sqrt{n_2}}{n_3} < 1$$

as required. We now show that $x_1 > 0$. Since $n_3 < 0$, it suffices to prove that $n_1 < \sqrt{n_2}$. Since $n_3 < 0$, we have $n_1^2 < n_2$ and so $-\sqrt{n_2} < n_1 < \sqrt{n_2}$. Thus $n_1 - \sqrt{n_2} < 0$ and so $x_1 > 0$. \square

3.3. Analysis of y_2 . We now consider the second root and show that it is inadmissible.

Lemma 3.7. *For all $a, b \in [0, 1]$, $y_2 > 1$.*

Proof. Since each summand of $m_1 + m_3$ from Equation (3.12) is positive, we have $m_1 + m_3 > 0$. Similarly $m_3 = 2a(-1 + a - b)b^2 < 0$ since $a - b < 1$. Since $m_1 + m_3 + \sqrt{n_2} > 0$, we have $-\frac{m_1 + \sqrt{n_2}}{m_3} > 1$, making y_2 inadmissible. \square

3.4. Summary of fixed points. Combining the results above yields

Theorem 3.8. *Let a, b be non-zero such that $b \neq a \pm \sqrt{a}$. Then f has exactly one valid non-trivial fixed point if and only if $b > \frac{1}{\sqrt{2}}(1 - a)$.*

Remark 3.9. *We may need to do a bit more analysis if $b = a + \sqrt{a}$, but this should be straightforward.*

4. GENERAL n WITH $b < (1 - a)/\sqrt{(n)}$: DYNAMICAL BEHAVIOR ANALYSIS (STEVE)

Below we analyze the dynamical behavior for any $n = 2$, provided that $b < (1 - a)/\sqrt{(n)}$. There are probably numerous ways of showing that, in this case, all iterates converge to the trivial fixed point. The following proof seems as good as any. It relies on the following lemma:

Lemma 4.1. *Let $a, b \in (0, 1)$ with $b < (1 - a)/\sqrt{(n)}$, and let $\lambda_1 \geq \lambda_2$ denote the eigenvalues of the matrix $\begin{pmatrix} a\alpha & nb\beta \\ b\gamma & a\delta \end{pmatrix}$, where $\alpha, \beta, \gamma, \delta \in (0, 1)$. Then $-1 < \lambda_1, \lambda_2 < 1$.*

Proof. The sum of the eigenvalues is the trace of the matrix (which is $a(\alpha + \delta)$), and the product of the eigenvalues is the determinant (which is $a^2\alpha\delta - nb^2\beta\gamma$). Thus the eigenvalues satisfy the characteristic equation

$$\lambda^2 - a(\alpha + \delta)\lambda + (a^2\alpha\delta - nb^2\beta\gamma). \quad (4.1)$$

The eigenvalues are therefore

$$\frac{a(\alpha + \delta) \pm \sqrt{a^2(\alpha + \delta)^2 - 4(a^2\alpha\delta - nb^2\beta\gamma)}}{2} = \frac{a(\alpha + \delta) \pm \sqrt{a^2(\alpha - \delta)^2 + 4nb^2\beta\gamma}}{2}. \quad (4.2)$$

As the discriminant is positive, the eigenvalues are real. Since $a(\alpha + \delta) \geq 0$, we have $|\lambda_2| \leq \lambda_1$, where

$$0 \leq \lambda_1 = \frac{a(\alpha + \delta) + \sqrt{a^2(\alpha - \delta)^2 + 4nb^2\beta\gamma}}{2}. \quad (4.3)$$

As $\beta\delta < 1$, $nb^2 \leq (1-a)^2$ and $\sqrt{u+v} \leq \sqrt{u} + \sqrt{v}$ for $u, v \geq 0$ we find

$$\begin{aligned} \lambda_1 &< \frac{a(\alpha + \delta) + \sqrt{a^2(\alpha - \delta)^2 + \sqrt{4(1-a)^2}}}{2} \\ &= \frac{a(\alpha + \delta) + a|\alpha - \delta| + 2(1-a)}{2} \\ &= \frac{2a \max(\alpha, \delta) + 2(1-a)}{2} \\ &= 1 - (1 - \max(\alpha, \delta))a < 1, \end{aligned} \tag{4.4}$$

where the last claim follows from $a, \alpha, \delta \in (0, 1)$. \square

Theorem 4.2. *Let $n \geq 2$. Assume $b \leq (1-a)/\sqrt{n}$. Then there is only one valid fixed point, the trivial fixed point (which may occur with multiplicity greater than 1). Further, iterates of any point converge to the trivial fixed point.*

Proof. We shall prove this by using the Mean Value Theorem and an eigenvalue analysis of the resulting matrix.

We have

$$f \left(\begin{pmatrix} u \\ v \end{pmatrix} \right) = \begin{pmatrix} 1 - (1-au)(1-bv)^n \\ 1 - (1-av)(1-bu) \end{pmatrix}. \tag{4.5}$$

Let

$$c(t) = (1-t) \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} x \\ y \end{pmatrix}, \quad c'(t) = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.6}$$

Thus $c(t)$ is the line connecting the trivial fixed point to $\begin{pmatrix} x \\ y \end{pmatrix}$, with $c(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

and $c(1) = \begin{pmatrix} x \\ y \end{pmatrix}$. Let

$$\mathcal{F}(t) = f(c(t)) = \begin{pmatrix} 1 - (1-atx)(1-bty)^n \\ 1 - (1-aty)(1-btx) \end{pmatrix}. \tag{4.7}$$

Then simple algebra (or the chain rule) yields

$$\mathcal{F}'(t) = \begin{pmatrix} a(1-bty)^n & nb(1-atx)(1-bty)^{n-1} \\ b(1-aty) & a(1-btx) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.8}$$

We now apply the one-dimensional chain rule twice, once to the x -coordinate function and once to the y -coordinate function. We find there are values t_1 and t_2 such that

$$f \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) - f \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} a(1-bt_1y)^n & nb(1-at_1x)(1-bt_1y)^{n-1} \\ b(1-at_2y) & a(1-bt_2x) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.9}$$

To see this, look at the x -coordinate of $\mathcal{F}(t)$: $h(t) = 1 - (1-atx)(1-bty)^n$. We have $h(1) - h(0) = h(1) = h'(t_1)(1-0)$ for some t_1 . As

$$\begin{aligned} h'(t_1) &= ax(1-bt_1y)^n + nby(1-at_1x)(1-bt_1y)^{n-1} \\ &= (a(1-bt_1y)^n, nb(1-at_1x)(1-bt_1y)^{n-1}) \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned} \tag{4.10}$$

the claim follows; a similar argument yields the claim for the y -coordinate (though we might have to use a different value of t , and thus denote the value arising from applying the Mean Value Theorem here by t_2).

We therefore have

$$\begin{aligned} f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \begin{pmatrix} a(1 - bt_1y)^n & nb(1 - at_1x)(1 - bt_1y)^{n-1} \\ b(1 - at_2y) & a(1 - bt_2x) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= A(a, b, x, y, t_1, t_2) \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned} \quad (4.11)$$

To show that f is a contraction mapping, it is enough to show that, for all a, b with $b \leq (1 - a)/\sqrt{n}$ and all $x, y \in [0, 1]$ that the eigenvalues of $A(a, b, x, y, t_1, t_2)$ are less than 1 in absolute value; however, this is exactly what Lemma 4.1 gives (note our assumptions imply that $\alpha = (1 - bt_1y)^n$ through $\delta = (1 - bt_2x)$ are all in $(0, 1)$). Let us denote $\lambda_{\max}(a, b)$ the maximum value of λ_1 for fixed a and b as we vary $t_1, t_2, x, y \in [0, 1]$. As we have a continuous function on a compact set, it attains its maximum and minimum. As λ_1 is always less than 1, so is the maximum. Here it is important that we allow ourselves to have $t_1, t_2 \in [0, 1]$, so that we have a closed and bounded set; it is immaterial (from a compactness point of view) that $a, b \in (0, 1)$ as they are fixed. It is important that $0 < a, b < 1$, as this ensures that $\alpha, \beta, \gamma, \delta < 1$ and so we have the strict inequalities claimed in Lemma 4.1. For any matrix M we have $\|Mv\| \leq |\lambda_{\max}||v|$; thus

$$\left\| f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \right\| \leq \lambda_{\max}(a, b) \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|; \quad (4.12)$$

as $\lambda_{\max}(a, b) < 1$ we have a contraction map. Therefore any non-zero $\begin{pmatrix} x \\ y \end{pmatrix}$ iterates to the trivial fixed point if $b < (1 - a)/\sqrt{n}$ and $n \geq 2$. In particular, the trivial fixed point is the only fixed point (if not, $A(a, b, x, y, t_1, t_2)v = v$ for v a fixed point, but we know $\|A(a, b, x, y, t_1, t_2)v\| < \|v\|$ if v is not the zero vector). \square

5. GENERAL n WITH $b > (1 - a)/\sqrt{n}$: DYNAMICAL BEHAVIOR ANALYSIS (STEVE)

5.1. Nature of the fixed points. We first analyze the nature of the fixed points. The following lemma will be useful.

Lemma 5.1. *Let $a, b \in (0, 1)$, and set*

$$A = \begin{pmatrix} a & nb \\ b & a \end{pmatrix}. \quad (5.1)$$

Then the eigenvalues of A are $a + b\sqrt{n}$, with corresponding eigenvector $\begin{pmatrix} \sqrt{n} \\ 1 \end{pmatrix}$, and $a - b\sqrt{n}$, with corresponding eigenvector $\begin{pmatrix} -\sqrt{n} \\ 1 \end{pmatrix}$. We may write any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{y}{2} + \frac{x}{2\sqrt{n}}\right) \begin{pmatrix} \sqrt{n} \\ 1 \end{pmatrix} + \left(\frac{y}{2} - \frac{x}{2\sqrt{n}}\right) \begin{pmatrix} -\sqrt{n} \\ 1 \end{pmatrix}. \quad (5.2)$$

If $b > (1 - a)/\sqrt{n}$ then $a + b\sqrt{n} > 1$.

Proof. The above claims follow by direct computation. It is easiest to write A as

$$A = aI + b\sqrt{n} \begin{pmatrix} 0 & \sqrt{n} \\ 1/\sqrt{n} & 0 \end{pmatrix} = aI + b\sqrt{n}B, \quad (5.3)$$

as the eigenvalues and eigenvectors of B are easily seen by inspection. \square

Remark 5.2. *The two eigenvectors are linearly independent, and thus a basis. Note that any vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ with positive coordinates will have a non-zero component in the $\begin{pmatrix} \sqrt{n} \\ 1 \end{pmatrix}$ direction. While we were able to explicitly compute the eigenvalues and eigenvectors here, we will not need the exact values of the eigenvectors below. From the Perron-Frobenius theorem we know that the largest (in absolute value) eigenvalue is positive and the corresponding eigenvector has all positive entries (because all entries in our matrix are positive).*

Theorem 5.3. *Assume $n \geq 2$, $a, b \in (0, 1)$ and $b > (1 - a)/\sqrt{n}$. Then there is a $\rho = \rho(a, b, n) > 0$ such that if $v = \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has $\|v\| \leq \rho$ then eventually an iterate of v by f is more than ρ units from the trivial fixed point. In other words, the trivial fixed point is repelling.*

Proof. We must show that if $\|v\|$ is sufficiently small then there is an m such that $\|f^{(m)}(v)\| > \|v\|$, where $f^{(2)}(v) = f(f(v))$ and so on.

We have

$$\begin{aligned} f \left(\begin{pmatrix} u \\ v \end{pmatrix} \right) &= \begin{pmatrix} 1 - (1 - au)(1 - bv)^n \\ 1 - (1 - av)(1 - bu) \end{pmatrix} \\ &= \begin{pmatrix} a & nb \\ b & a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + O_{a,b,n} \left(\begin{pmatrix} u^2 + v^2 \\ u^2 + v^2 \end{pmatrix} \right). \end{aligned} \quad (5.4)$$

In other words, there is some constant C (depending on n, a and b) such that the error in replacing f acting on $\begin{pmatrix} u \\ v \end{pmatrix}$ by the linear map $A = \begin{pmatrix} a & nb \\ b & a \end{pmatrix}$ acting on $\begin{pmatrix} u \\ v \end{pmatrix}$ is at most $C \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2$. Thus if $\begin{pmatrix} u \\ v \end{pmatrix}$ has small length, the error will be negligible.

To show that eventually an iterate of $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is further from the trivial fixed point than v , we argue as follows: we replace f by A , and since one of the eigenvalues is greater than one eventually an iterate will be further out. The argument is complicated by the need to do a careful book-keeping, as we must ensure that the error terms are negligible.

Let $\lambda_1 = a + b\sqrt{n} > 1$ and $\lambda_2 = a - b\sqrt{n}$ (note $|\lambda_2| < \lambda_1$ as we have assumed $a, b > 0$). We may write $\lambda = 1 + \eta$, with $0 < \eta < \sqrt{n}$. Our goal is to prove an equation of the form

$$f^{(m)}(v) = \lambda_1^m \left(\frac{y}{2} + \frac{x}{2\sqrt{n}} \right) \begin{pmatrix} \sqrt{n} \\ 1 \end{pmatrix} + \lambda_2^m \left(\frac{y}{2} - \frac{x}{2\sqrt{n}} \right) \begin{pmatrix} -\sqrt{n} \\ 1 \end{pmatrix} + \text{small}. \quad (5.5)$$

We often take m even, so that λ_2^m is non-negative. We may write $x = r \cos \theta$ and $y = r \sin \theta$, with $r \leq \rho$ (later we shall determine how large ρ may be).

We introduce some notation. By $E(z)$ we mean a vector $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ such that $|z_1|, |z_2| \leq z$. Let $v_0 = v$ and $v_{k+1} = f(v_k)$. Thus

$$v_1 = f(v_0) = Av_0 + E(Cr^2), \quad (5.6)$$

as $\|v_0\|^2 = r^2$; here $E(Cr^2)$ denotes our error vector, which has components at most Cr^2 . If $\|v_1\| > r$ then we have found an iterate which is further from the trivial fixed point, and we are done. If not, $\|v_1\| \leq r$.

Assume $\|v_1\| \leq r$. Then

$$v_2 = f(v_1) = Av_1 + E(Cr^2). \quad (5.7)$$

But $Av_1 = Av_0 + AE(Cr^2)$, with $E(Cr^2)$ denoting a vector with components at most Cr^2 . As the largest eigenvalue of A is λ_1 , we have $AE(Cr^2) = E(\lambda_1 Cr^2)$. Thus

$$v_2 = A^2 v_0 + E(\lambda_1 Cr^2 + Cr^2). \quad (5.8)$$

If $\|v_2\| > r$ we are done, so we assume $\|v_2\| \leq r$. Then

$$v_3 = f(v_2) = Av_2 + E(Cr^2). \quad (5.9)$$

But $Av_2 = A^3 v_0 + AE(\lambda_1 Cr^2 + Cr^2)$. As

$$AE(\lambda_1 Cr^2 + Cr^2) = E(\lambda_1^2 Cr^2 + \lambda_1 Cr^2), \quad (5.10)$$

we find

$$v_3 = A^3 v_0 + E(\lambda_1^2 Cr^2 + \lambda_1 Cr^2 + Cr^2). \quad (5.11)$$

If there is some m such that $\|v_m\| > r$ then we are done. If not, then for all m we have

$$v_m = A^m v_0 + E\left(\sum_{k=0}^{m-1} \lambda_1^k Cr^2\right) = A^m v_0 + E\left(\frac{\lambda_1^m - 1}{\lambda_1 - 1} \cdot Cr^2\right). \quad (5.12)$$

Using Lemma 5.1 (writing $v = v_0$ as a linear combination of the eigenvectors and applying A) yields

$$\begin{aligned} v_m &= \lambda_1^m \left(\frac{y}{2} + \frac{x}{2\sqrt{n}}\right) \begin{pmatrix} \sqrt{n} \\ 1 \end{pmatrix} + \lambda_2^m \left(\frac{y}{2} - \frac{x}{2\sqrt{n}}\right) \begin{pmatrix} -\sqrt{n} \\ 1 \end{pmatrix} \\ &\quad + E\left(\frac{\lambda_1^m - 1}{\lambda_1 - 1} \cdot Cr^2\right). \end{aligned} \quad (5.13)$$

We shall consider the case $x \geq y$; the other case follows similarly. Let m be the smallest even integer such that $\lambda_1^m \geq 10$; as $\lambda_1 < 1 + \sqrt{n} < 2\sqrt{n}$ we have for such m that $\lambda_1^m \leq 40n$. We consider the x -coordinate of v_m . As m is even and $x \geq y$ the contribution from

$$\lambda_1^m \left(\frac{y}{2} + \frac{x}{2\sqrt{n}}\right) \begin{pmatrix} \sqrt{n} \\ 1 \end{pmatrix} + \lambda_2^m \left(\frac{y}{2} - \frac{x}{2\sqrt{n}}\right) \begin{pmatrix} -\sqrt{n} \\ 1 \end{pmatrix} \quad (5.14)$$

is at least $\lambda_1^m \cdot \frac{x\sqrt{n}}{2\sqrt{n}} \geq 5x$; the contribution from $E\left(\frac{\lambda_1^m - 1}{\lambda_1 - 1} \cdot Cr^2\right)$ is at most $\frac{\lambda_1^m - 1}{\lambda_1 - 1} \cdot Cr^2 \leq \frac{\lambda_1^m}{\eta} \cdot Cr^2 \leq \frac{40Crn}{\eta} \cdot r$. By assumption, $r \leq \rho$. Let $\rho < \frac{\eta}{4000Cn}$. Then the x -coordinate

of v_m is at least $4x$ (since $x \geq y$, $x \geq r/\sqrt{2}$). Thus $\|v_m\|^2 \geq 16x^2 \geq 8(x^2 + y^2) = 8\|v\|^2 = 8r^2$, which contradicts $\|v_m\| \leq r$ for all m .

If instead $y \geq x$ then the same choices work, the only difference being that we now look at the y -coordinate. \square

Conjecture 5.4. *Let $n = 2$ and assume $a, b \in (0, 1)$ with $b > (1 - a)/\sqrt{n}$. The map f is a contraction map in a sufficiently small neighborhood of the unique non-trivial valid fixed point $v_f = \begin{pmatrix} x_f \\ y_f \end{pmatrix}$. Thus, if $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is sufficiently close to v_f , then the iterates of v converge to v_f .*

As of now, I can only prove this numerically. Unfortunately the linear approximation of f near the non-trivial valid fixed point v_f is a horrible mess, involving numerous complicated expressions of a and b . There are some things I can do to clean it up a bit, but not enough to get something which is algebraically transparent.

When $n = 2$ we have

$$y_f = \frac{bx_f}{1 - a + abx_f}, \quad x_f = \frac{(1 - a)y_f}{b(1 - ay_f)}. \quad (5.15)$$

Using $f\left(\begin{pmatrix} x_f \\ y_f \end{pmatrix}\right) = \begin{pmatrix} x_f \\ y_f \end{pmatrix}$ yields

$$(1 - bx_f) = \frac{1 - y_f}{1 - ay_f}, \quad (1 - by_f)^2 = \frac{1 - x_f}{1 - ax_f}. \quad (5.16)$$

These relations can help simplify some of the formulas; the problem is the formula for x_f in terms of a and b is a nightmare:

$$x_f = \frac{2a^3 + b^3 - 2a^2(2 + b) + a(2 + 2b - 2b^2) - b\sqrt{b^4 + 4a(1 - b)(a - 1 - b)^2}}{2ab(a^2 + b^2 - a(1 + 2b))}. \quad (5.17)$$

The resulting fixed point matrix is

$$A_f = \begin{pmatrix} a(1 - by_f)^2 & 2b(1 - ax_f)(1 - by_f) \\ b(1 - ay_f) & a(1 - bx_f) \end{pmatrix}. \quad (5.18)$$

We want to show the largest eigenvalue is less than 1 in absolute value when $b > (1 - a)/\sqrt{2}$.

We know that the critical line is $b = (1 - a)/\sqrt{2} = 1/\sqrt{2} - a/\sqrt{2}$. I've found a good way to numerically investigate the eigenvalues of A_f is study the eigenvalues along the line $b = (m - a)/\sqrt{2}$, with $1 < m < 1 + \sqrt{2}$. This gives us a family of parallel lines. For a given (valid) choice of m , we have $\max(0, m - \sqrt{2}) < a < 1$. Below (Figures 1 through 5) is an illustrative set of plots of the largest eigenvalue for 5 different choices of m .

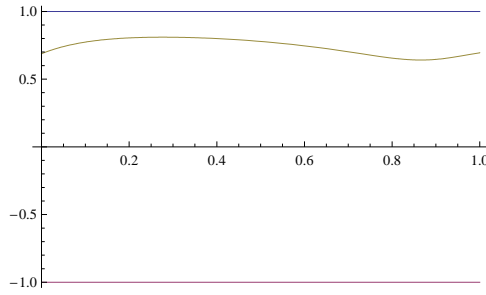


FIGURE 1. Distribution of the largest eigenvalue of A_f along the line $b = (m - a)/\sqrt{2}$, with $m = 1 + \sqrt{2}/6 \approx 1.2357$.

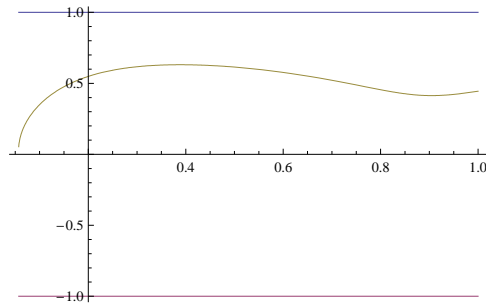


FIGURE 2. Distribution of the largest eigenvalue of A_f along the line $b = (m - a)/\sqrt{2}$, with $m = 1 + 2\sqrt{2}/6 \approx 1.4714$.

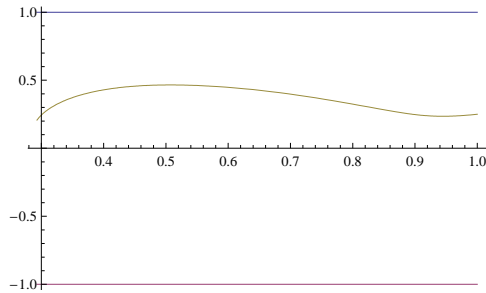


FIGURE 3. Distribution of the largest eigenvalue of A_f along the line $b = (m - a)/\sqrt{2}$, with $m = 1 + 3\sqrt{2}/6 \approx 1.7071$.

It is crucial that $m > 1$, as $m = 1$ leads to a coalescing of fixed points (i.e., we have the trivial fixed point with multiplicity two, and the third fixed point is not valid). In Figure 6 we plot the behavior of $1 - \lambda_1(a, 1 - \sqrt{2}/100)$, where $\lambda_1(a, b)$ is the largest eigenvalue of A_f . Note that the largest eigenvalue is very close to 1, but always less than 1, for this value of m .

Note in Figure 6 that λ_1 is small, especially for large a . This indicates that perhaps when a is close to 1 and $b = (m - a)/\sqrt{2}$ that there is a hope of proving the largest eigenvalue is strictly less than 1.

In fact, it is easy to show that if a and b are close to 1, then x_f is close to 1 as well (which immediately implies that y_f is also close to 1). This implies that the entries of

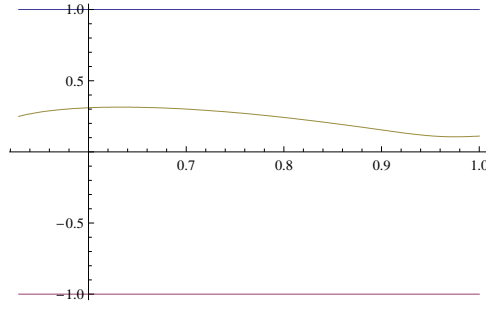


FIGURE 4. Distribution of the largest eigenvalue of A_f along the line $b = (m - a)/\sqrt{2}$, with $m = 1 + 4\sqrt{2}/6 \approx 1.9428$.

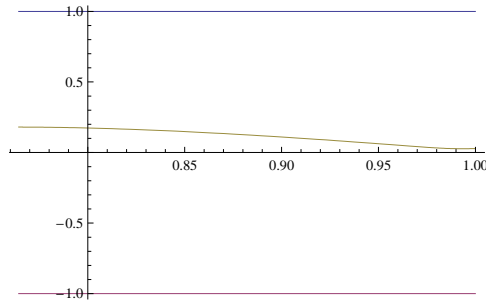


FIGURE 5. Distribution of the largest eigenvalue of A_f along the line $b = (m - a)/\sqrt{2}$, with $m = 1 + 5\sqrt{2}/6 \approx 2.1785$.

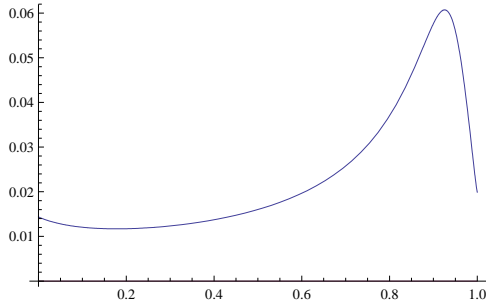


FIGURE 6. Distribution of 1 minus the largest eigenvalue of A_f along the line $b = (m - a)/\sqrt{2}$, with $m = 1 + \sqrt{2}/100 \approx 1.0141$.

A_f are all positive numbers close to 0. A simple calculation shows

$$\lambda_1(a, b) = \frac{((1 - by_f)^2 + (1 - ax_f)) a}{2} + \frac{\sqrt{((1 - by_f)^2 - (1 - ax_f)) a^2 + 8b^2(1 - by_f)(1 - ax_f)(1 - ay_f)}}{2}. \tag{5.19}$$

If a, b, x_f and y_f are all close to 1, then $\lambda_1(a, b)$ will be small. We have shown

Lemma 5.5. *Let $n = 2$, $a, b \in (0, 1)$ and assume $b > (1 - a)/\sqrt{2}$. Then if a and b are sufficiently large, then f is a contraction map near the non-trivial valid fixed point (i.e., the non-trivial valid fixed point is attracting).*

With some work we can determine how ‘close’ a and b need to be to 1.

5.2. Existence of a non-trivial, valid fixed point (new results: Steve). We show in this subsection that if $b > (1 - a)/\sqrt{n}$ then there is a unique, non-trivial valid fixed point when $a, b \in (0, 1)$. The proof involves looking at the intersection of two curves, one where the x -coordinate is unchanged under applying f , and one where the y -coordinate is unchanged after applying f . One of these curves is concave up, the other convex up. The proof is completed by the following lemma.

Lemma 5.6. *Let $h_1, h_2 : [0, 1] \rightarrow [0, 1]$ be twice continuously differentiable functions such that $h_1(x)$ is convex up, $h_2(x)$ is concave up, $h_1(0) = h_2(0) = 0$ and $h_1(x) \neq h_2(x)$ for $x > 0$ sufficiently small. Then for at most two choices of x do we have $h_1(x) = h_2(x)$.*

Proof. The claim is trivial if there is only one point of intersection, so assume there are at least two. Without loss of generality we may assume $p > 0$ is the first point above zero where h_1 and h_2 agree. Such a smallest point exists by continuity, as we have assumed $h_1(x) \neq h_2(x)$ for $x > 0$ sufficiently small; if there are infinitely many points x_n where they are equal, let $p = \liminf_n x_n > 0$. (We technically do not need to prove this – we could take any two points where the functions agree and show there cannot be a third point larger than the first two where the functions agree.)

Because $h_1(x)$ is convex up, $h_1'(x)$ is increasing. By the mean value theorem there is a point $c_1 \in (0, p)$ such that $h_1'(c_1) = (h_1(p) - h_1(0))/(p - 0) = h_1(p)/p$. As h_1' is increasing, we have $h_1'(p) > h_1'(c_1)$; further, $h_1'(x) > h_1'(c_1)$ for all $x \geq p$. As $h_2(x)$ is concave up, $h_2'(x)$ is decreasing. Again by the mean value theorem there is a point $c_2 \in (0, p)$ such that $h_2'(c_2) = ((h_2(p) - h_2(0))/(p - 0) = h_2(p)/p$. As h_2' is decreasing, we have $h_2'(p) < h_2'(c_2)$, and in fact $h_2'(x) < h_2'(c_2)$ for all $x \geq p$. But $h_1'(c_1) = h_2'(c_2)$ (since $h_1(p) = h_2(p)$), so $h_1'(x) > h_2'(x)$ for all $x \geq p$. Thus there cannot be another point of intersection after p . \square

Theorem 5.7. *Assume $a, b \in (0, 1)$, $b > (1 - a)/\sqrt{n}$ and $n \geq 2$. Then there exists a unique non-trivial, valid fixed point.*

Proof. We prove this through repeated applications of the Intermediate Value Theorem and continuity. Let

$$g \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} g_1(x, y) \\ g_2(x, y) \end{pmatrix} = f \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) - \begin{pmatrix} x \\ y \end{pmatrix}. \quad (5.20)$$

Note $\begin{pmatrix} x \\ y \end{pmatrix}$ is a fixed point if and only if $g \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = 0$.

We first look for partial fixed points, namely points where either the x or the y -coordinate is unchanged. These correspond to finding $\begin{pmatrix} x \\ y \end{pmatrix}$ with $g_1(x, y) = 0$ or $g_2(x, y) = 0$. We first analyze the set of pairs $(x, y) \in [0, 1]^2$ where $g_1(x, y) = 0$. We

have

$$g_1(x, y) = (1 - (1 - ax)(1 - by)^n) - x. \quad (5.21)$$

We immediately see that $g_1(0, 0) = 0$, $g_1(0, y) > 0$ for $y \in (0, 1]$, and $g_1(1, y) < 0$ for $y \in [0, 1]$. Thus by the Intermediate Value Theorem, for each $y \in (0, 1]$ there is a $\phi_1(y)$ such that $g_1(\phi_1(y), y) = 0$ and $\phi_1(y) \in [0, 1]$. It is easy to see that $\phi_1(y)$ is a continuous function of y ; in fact,

$$\begin{aligned} \phi_1(y) &= \frac{1 - (1 - by)^n}{1 - a(1 - by)^n} \\ \phi_1'(y) &= \frac{nb(1 - a)(1 - by)^{n-1}}{(1 - a(1 - by)^n)^2}. \end{aligned} \quad (5.22)$$

Note $\phi_1(y) \in [0, 1]$: it is clearly positive, and $\frac{1-c}{1-ac} > 1$ for $c > 0$ only when $a > 1$. As $a, b \in (0, 1)$, $\phi_1'(y) > 0$. Thus $\phi_1(y)$ is strictly increasing, and $\phi_1(0) = 0$. Further, we have for small y that $\phi_1(y) \approx \frac{nb}{1-a}y$. To see this, we note $(1 - by)^n = 1 - nby + O(y^2)$ and substitute into (5.22). To aid in the analysis below, it is more convenient to re-write this as $y \approx \frac{1-a}{nb}x$ (as $\phi_1'(y) > 0$ we may use the inverse function theorem to write y as a function of x).

We analyze $g_2(x, y) = 0$ similarly. We find

$$g_2(x, y) = (1 - (1 - ay)(1 - bx)) - y = 0. \quad (5.23)$$

Note $g_2(0, 0) = 0$, $g_2(x, 0) > 0$ for $x \in (0, 1]$, and $g_2(x, 1) < 0$ for $x \in [0, 1]$. Solving yields

$$y = \phi_2(x) = \frac{bx}{1 - a + abx}. \quad (5.24)$$

This is clearly continuously differentiable, and

$$\phi_2'(x) = \frac{b(1 - a)}{(1 - a + abx)^2} > 0. \quad (5.25)$$

Thus $\phi_2(x)$ is an increasing function of x . Further, for small x we have $y \approx \frac{b}{1-a}x$.

We now use the assumption that $b > (1 - a)/\sqrt{n}$. Near the origin, $\phi_1(y)$ looks like the line $y = \frac{1-a}{nb}x$, while near the origin $\phi_2(x)$ looks like the line $y = \frac{b}{1-a}x$. If $\frac{1-a}{nb} < \frac{b}{1-a}$ then $\phi_2(x)$ is above $\phi_1(y)$ near the origin. Cross multiplying shows that this condition is equivalent to $b^2 > (1 - a)/n$, or $b > (1 - a)/\sqrt{n}$. Thus, for $a, b \in (0, 1)$ and $b > (1 - a)/\sqrt{n}$, the two curves $x = \phi_1(y)$ and $y = \phi_2(x)$ have at least two intersections in $[0, 1]^2$; one is the trivial fixed point while the other is a non-trivial, valid fixed point. The existence of the second point of intersection follows from the intermediate value theorem (near the origin $y = \phi_2(x)$ is above $x = \phi_1(y)$; however, as $x \rightarrow 1$ we have $\phi_2(x)$ tends to a number strictly less than 1. Thus the curve $y = \phi_2(x)$ hits the line $x = 1$ below $(1, 1)$. Similarly the curve $x = \phi_1(y)$ hits the line $y = 1$ to the left of $(1, 1)$. Thus the two curves flip as to which is above the other, implying that there must be one point where the two curves are equal. This point is clearly a fixed point.

We now show there are only two intersections (i.e., there is a unique, non-trivial valid fixed point). The proof follows from showing that $y = \phi_2(x)$ is concave up (concave increasing) and $x = \phi_1(y)$ is convex up (convex increasing). There are already two

points of intersection, and by Lemma 5.6 there can be at most two points of intersection. Straightforward differentiation and some algebra gives

$$\begin{aligned}\phi_2''(x) &= \frac{-2ab^2(1-a)}{(1-a+abx)^2} < 0 \\ \phi_1''(y) &= -\frac{b^2n(1-a)(1-by)^{n-2} \cdot (n-1+a(1-by)^n+a(n+1)(1-by)^n)}{(1-a(1-by)^n)^3} < 0.\end{aligned}\tag{5.26}$$

Thus $y = \phi_2''(x)$ is concave up (since the second derivative is always negative and the first derivative is always positive: compare this to the standard parabola $y = -x^2$ when $x < 0$). As a function of y , $x = \phi_1(y)$ is also concave up (since its first derivative is positive and its second derivative is negative); however, we are interested in $y = \phi_1^{-1}(x)$ (the inverse function exists because the first derivative is positive). If $\phi_1(y)$ is concave up as a function of y then $\phi_1^{-1}(x)$ is convex up as a function of x . This follows because we are basically reflecting about the $x = y$ line, and this switches us from concave to convex (the function is obviously still increasing). The claim now follows from Lemma 5.6. \square

In Figures 7 and 8 we plot $x = \phi_1(y)$ and $y = \phi_2(x)$ for $a = .4$, $b = .5$ and $n \in \{2, 5\}$. As should be the case, the intersections of the two curves correspond to the fixed points (obtained by solving $f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$).

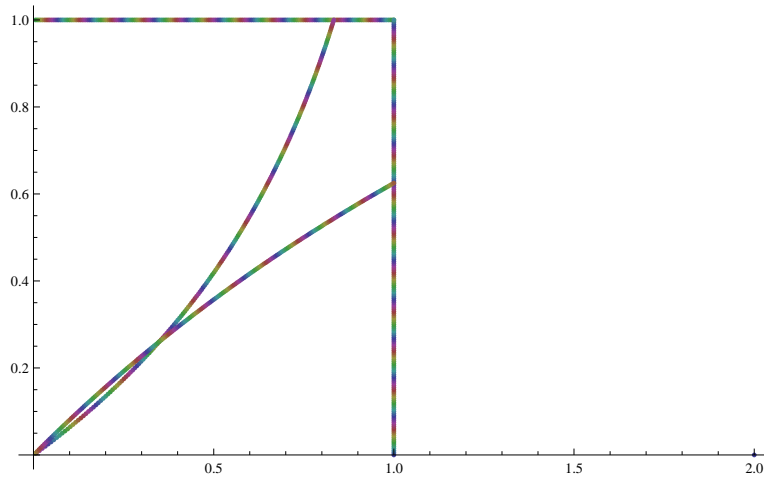


FIGURE 7. Plot of $x = \phi_1(y)$ and $y = \phi_2(x)$ for $a = .4$, $b = .5$ and $n = 2$. The fixed point is $(.350, .261)$.

Remark 5.8 (VERY TENTATIVE). *We can try and use the above argument to show the fixed point is attracting, at least locally. The two curves $x = \phi_1(y)$ and $y = \phi_2(x)$ break $[0, 1]^2$ into four pieces.*

- (1) *The region under $\phi_2(x)$ and to the left of $\phi_1(y)$: in this region the effect of applying f is to increase the values of both x and y .*

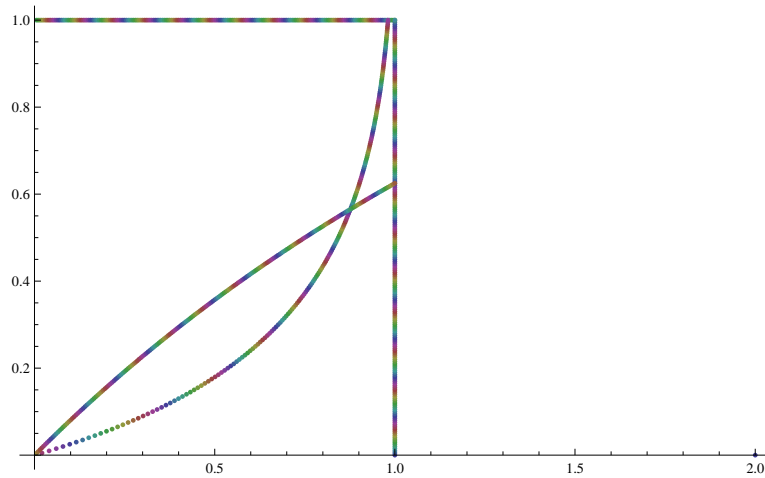


FIGURE 8. Plot of $x = \phi_1(y)$ and $y = \phi_2(x)$ for $a = .4$, $b = .5$ and $n = 5$. The fixed point is $(.877, .565)$.

- (2) *The region under $\phi_2(x)$ and to the right of $\phi_1(y)$: in this region the effect of applying f is to decrease the value of x and increase the value of y .*
- (3) *The region above $\phi_2(x)$ and to the right of $\phi_1(t)$: in this region the effect of applying f is to decrease both x and y .*
- (4) *The region above $\phi_2(x)$ and to the left of $\phi_1(y)$: in this region the effect of applying f is to increase x and decrease y .*

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E-mail address: aryehk@wisdom.weizmann.ac.il

INSERT

E-mail address: sjmiller@math.brown.edu

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912

E-mail address: aroy@cs.bc.edu

DEPARTMENT OF COMPUTER SCIENCE, BOSTON COLLEGE, CHESTNUT HILL, MA 02467