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On the Barban-Davenport-Halberstam theorem: VIII

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1. Introduction

We return yet again to the subject of the distribution of the differences

$$E(x;a,k)=\theta(x;a,k)-\frac{x}{\phi(k)}$$

for relatively prime values of a, k, where

$$\theta(x;a,k)=\sum_{\substack{p\leqq x\\ p\equiv a,\bmod k}}\log p$$

as usual. Having mainly directed our attention in the earlier articles of this series (denoted in what follows by the Roman numeral indicating their place in our researches) to questions associated with either

$$\sum_{\substack{0<a\leqq k\\ (a,k)=1}}E^2(x;a,k)$$

or the Barban-Davenport-Halberstam moments

$$\sum_{k\leqq Q}\sum_{\substack{0<a\leqq k\\ (a,k)=1}}E^2(x;a,k)\,,$$

we now pass on to the study of the third moment

$$S(Q)=\sum_{k\leqq Q}\phi(k)\sum_{\substack{0<a\leqq k\\ (a,k)=1}}E^3(x;a,k) \tag{1}$$

and prove that, for $Q=o(x/\log x)$ and any positive constant A,

$$S(Q)=o\left(Q^{\frac{3}{2}}x^{\frac{3}{2}}\log^{\frac{3}{2}}x\right)+O\left(\frac{x^3}{\log^A x}\right) \tag{2}$$

in the spirit of the announcement made at the end of the immediately preceding VII. We thus substantiate a further result that is consistent with the expectation that usually

$$E(x;a,k) = O\left(\frac{x^{\frac{1}{2}}\log^{\frac{1}{2}} x}{\phi^{\frac{1}{2}}(k)}\right) \quad ((a,k)=1)$$

for values of k almost up to $x/\log x$, providing in addition some considerable confirmation of our belief that $E(x;a,k)$ is symmetrically distributed about its effectively zero mean. We also obtain the counterpart of (2) that is valid for the complementary range

$$x/\log x \leqq Q \leqq x\,,$$

albeit this is of less significance in the theory we are illustrating.

Although our method has a genesis somewhat similar to that of our proofs of the Barban-Montgomery theorem in I and II, the main body of our investigation introduces features and complications that were absent from previous contributions to this series. In particular, the most important aspect of the analysis stems from a special variant of Vinogradov's theorem concerning the representation of zero by ternary linear combinations of primes, the application of which requires us to pursue a narrow track very precisely in order to avoid inexactitudes that would vitiate our objective. The essential points of importance being necessarily somewhat obscured by the complexity of the workings, we can perhaps admit that there were occasions in the investigation when like Dante (*Inferno*) we found ourselves

"..... in a gloomy wood, astray
Gone from the path direct .....".

In fact, having dissected the right-hand side of (1) into sums involving various combinations of $\theta(x;a,k)$ and $x/\phi(k)$, we identify the main difficulty in the need to evaluate these items so accurately that the explicit terms in their determinations annihilate themselves to leave only the right-hand side of (2); here it is necessary to take into account not only main terms of approximate sizes $x^3\log^a x$ but also secondary exact terms of approximate size $Qx^2\log^\beta x$. But it is best to defer discussion of these and other obstacles athwart our path until we reach and confront them.

The choice of the multiplier affecting the inner sum in (1) is not particularly critical and we have therefore taken it to be $\phi(k)$ in the interests of simplicity. Equally well, we could have used either $\phi^{\frac{1}{2}}(k)$ as in VII or k, although at the expense of a little further complication in the proof. Yet, increasingly as the order of the moments of this type becomes larger, it is desirable to use weights that are sufficiently large for the contribution of the larger values of k to predominate.

The assumption of the extended Riemann hypothesis for Dirichlet's L-functions enables one to obtain a greatly improved form of our result in which a sharper version of (2) is valid for smaller values of Q, thus shedding further light on the distribution of $E(x;a,k)$ for smaller values of k. But the investigation of this development must await a later paper in this series, since otherwise we would exceed the limits that have been set for the present occasion.

2. Notation

Owing to the length and complexity of the memoir it is not practicable to lay down a completely consistent notation. However, the meaning of all symbols should be clear from their context in the light of the following guide.

The letters p, p_1, p_2, p_3, ϖ denote (positive) prime numbers; $m, n, \ell, \ell_1, \ell_2, \ell_3, \ell', \ell_1', \ell_2', \ell_3'$, are positive integers; d, δ are positive integers that play various interconnected rôles; d', Δ, Δ' are also positive integers that arise from d, δ in one of their incarnations.

The letters C_i are positive constants that are explicitly defined; the letters B_i are also specific constants whose precise values are immaterial to the investigation; A' is a positive constant whose value is not necessarily the same at each occurrence; $A, A_1, \ldots, A_{12}$ are positive absolute constants whose connection with each other will be plain from the text and, in particular, from § 5; ε is an arbitrarily small positive constant that is not necessarily the same on all occasions.

The constants implied by the O-notation depend at most on A', A, A_i, and ε in a manner that is clear from the text; ultimately, however, they depend only on A. The function $\Gamma(m)$ is of an arithmetical nature save when it is designated to be the usual (Eulerian) gamma function. As usual, (a, b) and $[a, b]$ respectively denote the positive highest common factor and least common multiple of a, b when these are defined.

3. Initial decomposition of the sum

Expanding the notation in (1) by writing

$$S(x, Q) = \sum_{k \leqq Q} \phi(k) \sum_{\substack{0 < a \leqq k \\ (a,k)=1}} E^3(x; a, k),$$

we fix the positive absolute constant A that is to appear in (2) and first consider the case where $Q \leqq x \log^{-A_1} x$ and A_1 is a suitable absolute constant such that

$$A_1 \geqq A + 2. \tag{3}$$

Then, since in this instance

$$E(x; a, k) \leqq \frac{x}{\phi(k)} + \log x \sum_{\substack{n \leqq x \\ n \equiv a, \bmod k}} 1 = O\left(\frac{x \log x}{\phi(k)}\right),$$

we have immediately that

$$S(x, Q) = O\left(x \log x \sum_{k \leqq Q} \sum_{\substack{0 < a \leqq k \\ (a,k)=1}} E^2(x; a, k)\right) = O\left(\frac{x}{\log^A x}\right) \tag{4}$$

by Gallagher's form of Theorem A in I. In the contrary case that remains to be considered, we let

$$x \log^{-A_1} x = Q_1 \leqq Q \leqq x \tag{5}$$

and then write

$$S(x; Q_1, Q) = S(x, Q) - S(x, Q_1)$$

so that

$$S(x, Q) = S(x; Q_1, Q) + O\left(\frac{x}{\log^A x}\right) \tag{6}$$

in virtue of (4) above.

Next, decomposing $S(x; Q_1, Q)$ with the aid of the simple identity

$$(u - v)^3 = u^3 - 3(u - v)^2 v - 3(u - v) v^2 - v^3,$$

we obtain

$$\begin{aligned} S(x; Q_1, Q) &= \sum_{Q_1 < k \leqq Q} \phi(k) \sum_{\substack{0 < a \leqq k \\ (a,k)=1}} \theta^3(x; a, k) - 3x \sum_{Q_1 < k \leqq Q} \sum_{\substack{0 < a \leqq k \\ (a,k)=1}} E^2(x; a, k) \\ &\quad - 3x^2 \sum_{Q_1 < k \leqq Q} \frac{1}{\phi(k)} \sum_{\substack{0 < a \leqq k \\ (a,k)=1}} E(x; a, k) - x^3 \sum_{Q_1 < k \leqq Q} \frac{1}{\phi(k)} \\ &= S_1(x; Q_1, Q) - 3x S_2(x; Q_1, Q) - 3x^2 S_3(x; Q_1, Q) - x^3 \sum_{Q_1 < k \leqq Q} \frac{1}{\phi(k)}, \quad \text{say}, \end{aligned} \tag{7}$$

the last three terms in which will not delay us for long. Indeed the prime number theorem and Lemma 1 in I immediately give

$$\begin{aligned} S_3(x; Q_1, Q) &= \sum_{Q_1 < k \leqq Q} \frac{1}{\phi(k)} \{\theta(x) + O(\log k) - x\} = O(x \log x e^{-A' \sqrt{\log x}}) \\ &= O\left(\frac{x}{\log^A x}\right) \end{aligned} \tag{8}$$

and

$$\begin{aligned} \sum_{Q_1 < k \leqq Q} \frac{1}{\phi(k)} &= \frac{\zeta(2)\zeta(3)}{\zeta(6)} \log \frac{Q}{Q_1} + O\left(\frac{\log Q_1}{Q_1}\right) \\ &= \frac{\zeta(2)\zeta(3)}{\zeta(6)} \log \frac{Q}{Q_1} + O\left(\frac{1}{\log^A x}\right), \end{aligned} \tag{9}$$

respectively, whereas the sum $S_2(x; Q_1, Q)$ is estimated through an improvement in the form of the Barban-Montgomery theorem that was derived in I.

This refinement stems from the replacement of the first part of Lemma 1 in I by the more accurate

Lemma 1. *For $\xi \geqq 1$, we have*

$$\sum_{\ell < \xi} \left(1 - \frac{\ell}{\xi}\right)^2 \frac{1}{\phi(\ell)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \log \xi + C_1 + \frac{\log \xi}{\xi} + \frac{C_2}{\xi} + O\left(\frac{e^{-A'\sqrt{\log 2\xi}}}{\xi^{\frac{3}{2}}}\right),$$

where

$$C_2 = \frac{\zeta'(0)}{\zeta(0)} + \gamma + \sum_p \frac{\log p}{p(p-1)}. \tag{10}$$

All that is needed being a careful reappraisal of the proof of the previous result in I, we first observe that the integrand in the contour integral there is equal to

$$\left(\frac{1}{(s+1)^2} + \frac{\gamma}{s+1} + \cdots\right) \frac{\zeta(s+1)\, h(s+1)\, \xi^s}{s(s+2)}$$

in the neighbourhood of $s = -1$, whence with the aid of logarithmic differentiation we obtain

$$R_2 = -\frac{\zeta(0)\, h(0)}{\xi}\left(\log \xi + \frac{\zeta'(0)}{\zeta(0)} + \frac{h'(0)}{h(0)} + \gamma\right)$$

and confirm the above stated value of C_2 because $\zeta(0) = -\dfrac{1}{2}$ and

$$\frac{h'(0)}{h(0)} = \sum_p \frac{\log p}{p(p-1)}$$

by a simple calculation. Secondly, a typical factor in the infinite product for $h(s)$ equals

$$\begin{aligned}
1 + &\left(1 - \frac{1}{p}\right)^{-1} \frac{1}{p^{s+2}} - \left(1 - \frac{1}{p}\right)^{-1} \frac{1}{p^{2s+2}} \\
&= 1 - \frac{1}{p^{2s+2}} + \frac{1}{p^{s+2}}\left(1 - \frac{1}{p}\right)^{-1}\left(1 - \frac{1}{p^{s+1}}\right) \\
&= \left(1 - \frac{1}{p^{2s+2}}\right)\left\{1 + \frac{1}{p^{s+2}}\left(1 - \frac{1}{p}\right)^{-1}\left(1 + \frac{1}{p^{s+1}}\right)^{-1}\right\}
\end{aligned}$$

with the implication that

$$h(s) = \frac{h_1(s)}{\zeta(2s+2)},$$

where $h_1(s)$ is regular and bounded for $\sigma > -1 + \eta$. Hence, utilizing features of $\zeta(s)$ such as its zero-free region and its order of magnitude, we see in the customary manner that the residual contour integral is

$$O\left(\xi^{-\frac{3}{2}} e^{-A'\sqrt{\log 2\xi}}\right)$$

and complete the proof of the revised lemma.

Observe now that our sum $S_2(x; Q_1, Q)$ is the sum $G(x; Q_1, Q_2)$ that is estimated in I after its appearance in equation (2) therein. Hence, using the revised lemma in the place of the original, we retrace the previous proof and obtain

$$(11) \qquad S_2(x; Q_1, Q) = Qx \log Q - (C_2 + 1) Qx + O\left(Q^{\frac{3}{2}} x^{\frac{1}{2}} e^{-A'\sqrt{\log 2x/Q}}\right) + O\left(\frac{x^2}{\log^A x}\right),$$

where it is to be borne in mind that the value of Q_1 is actually now smaller than in I.

We note that the use of the multiplier $\phi(k)$ in (1) instead of k enabled the sum $S_2(x; Q_1 Q)$ to be quickly treated by appealing to I; had the factor k been used more work would have been needed at this stage although some of the later analysis would have been simplified.

In summation of what has so far been learnt, we conclude from (6), (7), (8), (9) and (11) that

$$(12) \quad S(x, Q) = S_1(x; Q_1, Q) - \frac{\zeta(2)\zeta(3)}{\zeta(6)} x^3 \log \frac{Q}{Q_1} - 3Qx^2 \log Q + 3(C_2 + 1) Qx^2 + O\left(Q^{\frac{3}{2}} x^{\frac{3}{2}} e^{-A'\sqrt{\log 2x/Q}}\right) + O\left(\frac{x^3}{\log^A x}\right),$$

the stage having been now cleared for the treatment of the most important constituent $S_1(x; Q_1, Q)$ of $S(x, Q)$.

4. Dissection of $S_1^*(x; Q_1 Q)$ and the estimations of two of its constituents

The factor $\phi(k)$ affecting the inner sum in $S_1(x; Q_1, Q)$ creates difficulties in the analysis that are abated in the treatment of the associated sum

$$(13) \qquad S_1^*(x; Q_1, Q) = \sum_{Q_1 < k \leq Q} \frac{\phi(k)}{k} \sum_{\substack{0 < a \leq k \\ (a,k)=1}} \theta^3(x; a, k),$$

which therefore is the object of our study until we revert at the end to the original sum by partial summation.

The inner sum is

$$\sum_{\substack{0 < a \leq k \\ (a,k)=1}} \Bigg(\sum_{\substack{p \leq x \\ p \equiv a, \bmod k}} \log p \Bigg)^3 = \sum_{\substack{p_1 \equiv p_2 \equiv p_3, \bmod k \\ p_1, p_2, p_3 \leq x;\, (p_1 p_2 p_3, k) = 1}} \log p_1 \log p_2 \log p_3,$$

in the second term of which the first condition of summation implies the third when not all of p_1, p_2, p_3 are equal. Hence we obtain the equivalent determination

$$\left(6\sum_{\substack{p_1\equiv p_2\equiv p_3,\,\mathrm{mod}\,k\\ p_2<p_3<p_1\leqq x}}+3\sum_{\substack{p_1\equiv p_3,\,\mathrm{mod}\,k\\ p_2=p_3<p_1\leqq x}}+3\sum_{\substack{p_1\equiv p_2,\,\mathrm{mod}\,k\\ p_2<p_3=p_1\leqq x}}+\sum_{\substack{p_1=p_2=p_3\leqq x\\ (p_1p_2p_3,k)=1}}\right)\log p_1\log p_2\log p_3$$

$$=6\sum_{\substack{p_2<p_3<p_1\leqq x\\ p_1\equiv p_2\equiv p_3,\,\mathrm{mod}\,k}}\log p_1\log p_2\log p_3+3\sum_{\substack{p<p'\leqq x\\ p\equiv p',\,\mathrm{mod}\,k}}\log^2 p\log p'$$

$$+3\sum_{\substack{p<p'\leqq x\\ p\equiv p',\,\mathrm{mod}\,k}}\log p\log^2 p'+\sum_{\substack{p\leqq x\\ p\nmid k}}\log^3 p\,, \tag{14}$$

the individual contributions from which to $S_1^*(x;Q_1,Q)$ are denoted according to their positions by $6\,S_4^*(x;Q_1,Q)$, $3\,S_5^*(x;Q_1,Q)$, $3\,S_6^*(x;Q_1,Q)$, and $S_7^*(x;Q_1,Q)$ so that altogether

$$S_1^*(x;Q_1,Q) \tag{15}$$
$$=6\,S_4^*(x;Q_1,Q)+3\,S_5^*(x;Q_1,Q)+3\,S_6^*(x;Q_1,Q)+S_7^*(x;Q_1,Q)\,.$$

Once more the treatment of sums proceeds according to ascending order of difficulty. The inner sum in $S_7^*(x;Q_1,Q)$ is

$$\sum_{p\leqq x}\log^3 p+O\left(\log^2 x\sum_{p\mid k}\log p\right)=\sum_{p\leqq x}\log^3 p+O(\log^3 x) \tag{16}$$
$$=\int_{3/2}^{x}\log^2 t\,d\theta(t)+O(\log^3 x)$$
$$=x\log^2 x-2x\log x+2x+O\left(\frac{x}{\log^A x}\right)$$

by the prime number theorem. Therefore, invoking the well-known asymptotic formula

$$\sum_{k\leqq u}\frac{\phi(k)}{k}=\frac{u}{\zeta(2)}+O(\log 2u)\,,$$

we infer that

$$S_7^*(x;Q_1,Q)=\{x\log^2 x+O(x\log x)\}\sum_{Q_1<k\leqq Q}\frac{\phi(k)}{k} \tag{17}$$
$$=\frac{1}{\zeta(2)}\{Q-Q_1+O(\log x)\}\{x\log^2 x+O(x\log x)\}$$
$$=\frac{1}{\zeta(2)}Qx\log^2 x+O(Qx\log x)+O\left(\frac{x}{\log^A x}\right)$$

in view of (3) and (5). As we shall see, however, this implicit contribution to $S(x,Q)$ will only be of importance when Q lies in the exceptional range for which $Q\geqq x/\log x$.

To prepare the remaining sums $S_i^*(x;Q_1,Q)$ for their examination, we note their definition is only contingent on the condition $Q_1 \leqq Q \leqq x$ so that we may concentrate on the sums

$$J_i(x,Q) = S_i^*(x;Q,x) \quad (i=4,5,6) \tag{18}$$

for

$$Q \geqq x\log^{-A_1} x \tag{19}$$

because

$$S_i^*(x;Q_1,Q) = J_i(x,Q_1) - J_i(x,Q)\,. \tag{20}$$

The sums $S_5^*(x;Q_1,Q)$ and $S_6^*(x;Q_1,Q)$ are treated separately via (14) and then are amalgamated so that their sum can be assessed. First, considering the conditions of summation for $J_6(x,Q)$ that are inherent in (13), (14), (18), and (19) and letting ℓ denote generally a positive integer, we have

$$J_6(x,Q) = \sum_{\substack{p'-p=\ell k\\ p'\leqq x\\ Q<k\leqq x}} \log^2 p' \log p\, \frac{\phi(k)}{k}, \tag{21}$$

wherein the condition $k>Q$ implies that $\ell < x/Q$ and $p < p' - \ell Q$. Therefore

$$\begin{aligned} J_6(x,Q) &= \sum_{\substack{p'-p=\ell k\\ p'\leqq x;\,\ell<x/Q\\ p<p'-\ell Q}} \log^2 p' \log p \sum_{d\mid k} \frac{\mu(d)}{d} \\ &= \sum_{\ell<x/Q}\ \sum_{d\leqq x} \frac{\mu(d)}{d} \sum_{p'\leqq x} \log^2 p' \sum_{\substack{p<p'-\ell Q\\ p\equiv p',\ \mathrm{mod}\,\ell d}} \log p\,, \end{aligned} \tag{22}$$

in which, the innermost sum being

$$\frac{p'-\ell Q}{\phi(\ell d)} + O\left(\frac{x}{\log^{2A_1} x}\right)$$

when $p' \nmid \ell d$ but zero otherwise, we see that the double iterated sum over primes is

$$\begin{aligned} &\frac{1}{\phi(\ell d)} \sum_{\substack{p'\leqq x\\ p'\nmid \ell d}} (p'-\ell Q)\log^2 p' + O\left(\frac{x^2}{\log^{2A_1-1} x}\right) \\ &= \frac{1}{\phi(\ell d)}\left((x-\ell Q)\sum_{p'\leqq x}\log^2 p' - \sum_{p'\leqq x}(x-p')\log^2 p'\right) + O\left(\frac{x^2}{\log^{2A_1-1} x}\right). \end{aligned} \tag{23}$$

But, much as in (16),

$$\sum_{p'\leqq u} \log^2 p' = u\log u - u + O\left(\frac{u}{\log^{2A_1-1} u}\right)$$

and

$$\sum_{p'\leqq u}(u-p')\log^2 p' = \int_0^u \sum_{p'\leqq t}\log^2 p'\,dt \tag{24}$$

$$= \frac{1}{2}u^2\log u - \frac{3}{4}u^2 + O\left(\frac{(u+1)^2}{\log^{2A_1-1}(u+2)}\right),$$

for any $u>0$, whence (23) is equal to

$$\frac{1}{\phi(\ell d)}\left((x-\ell Q)(x\log x - x) - \left(\frac{1}{2}x^2\log x - \frac{3}{4}x^2\right)\right) + O\left(\frac{x^2}{\log^{2A_1-1}x}\right). \tag{25}$$

Also, since $J_5(x,Q)$ arises when p and p' are interchanged in the conditions of summation on the right side of (21), the substitution of the demands on the sizes of p and p' in (22) by $p'+\ell Q<p\leqq x$ and $p'<x-\ell Q$ yields a formula for $J_5(x,Q)$, in which now the right side has an innermost sum

$$\sum_{\substack{p'+\ell Q<p\leqq x\\ p\equiv p',\,\mathrm{mod}\,\ell d}}\log p = \frac{x-\ell Q-p'}{\phi(\ell d)} + O\left(\frac{x}{\log^{2A_1}x}\right)$$

for any p' not dividing ℓd. The double iterated sum over p, p' being therefore now

$$\frac{1}{\phi(\ell d)}\sum_{\substack{p'<x-\ell Q\\ p'\nmid \ell d}}(x-\ell Q-p')\log^2 p' + O\left(\frac{x^2}{\log^{2A-1}x}\right)$$

$$= \frac{1}{\phi(\ell d)}\left(\frac{1}{2}(x-\ell Q)^2\log(x-\ell Q) - \frac{3}{4}(x-\ell Q)^2\right) + O\left(\frac{x^2}{\log^{2A_1-1}x}\right)$$

by (24), we combine this with (25) and deduce that

$$J_5(x,Q)+J_6(x,Q) = \sum_{\ell<x/Q}\left((x-\ell Q)(x\log x - x) - \left(\frac{1}{2}x^2\log x - \frac{3}{4}x^2\right)\right.$$

$$\left. + \frac{1}{2}(x-\ell Q)^2\log(x-\ell Q) - \frac{3}{4}(x-\ell Q)^2\right)\sum_{d\leqq x}\frac{\mu(d)}{d\phi(\ell d)}$$

$$+ O\left(\frac{x^2}{\log^{2A_1-1}x}\sum_{\ell\leqq\log^{A_1}x}\ \sum_{d\leqq x}\frac{1}{d}\right)$$

$$= \sum_{\ell<x/Q}\left((x-\ell Q)(x\log x - x) - \left(\frac{1}{2}x^2\log x - \frac{3}{4}x^2\right) + \frac{1}{2}(x-\ell Q)^2\log(x-\ell Q)\right.$$

$$\left. - \frac{3}{4}(x-\ell Q)^2\right)\sum_{d=1}^{\infty}\frac{\mu(d)}{d\phi(\ell d)}$$

$$+ O\left(x^2\log x\sum_{\ell\leqq x}\frac{1}{\phi(\ell)}\sum_{d>x}\frac{1}{d\phi(d)}\right) + O\left(\frac{x^2}{\log^{A_1-2}x}\right)$$

$$= \sum_{\ell < x/Q} \Bigg((x - \ell Q)(x \log x - x) - \left(\frac{1}{2} x^2 \log x - \frac{3}{4} x^2 \right) + \frac{1}{2} (x - \ell Q)^2 \log (x - \ell Q)$$

$$- \frac{3}{4} (x - \ell Q)^2 \Bigg) \sum_{d=1}^{\infty} \frac{\mu(d)}{d \phi(\ell d)} + O\left(\frac{x^2}{\log^A x} \right)$$

in the light of (3) and (5). In this, by Euler's theorem, the series over d is

$$\prod_{p \nmid \ell} \left(1 - \frac{1}{p(p-1)} \right) \prod_{p^a \| \ell} \left(\frac{1}{\phi(p^a)} - \frac{1}{p^2 \phi(p^a)} \right)$$

$$= \frac{1}{\phi(\ell)} \prod_p \left(1 - \frac{1}{p(p-1)} \right) \prod_{p \mid \ell} \left(1 - \frac{1}{p(p-1)} \right)^{-1} \left(1 - \frac{1}{p^2} \right) = \frac{C_3 \psi_1(\ell)}{\ell},$$

where

$$C_3 = \prod_p \left(1 - \frac{1}{p(p-1)} \right) \tag{26}$$

and

$$\psi_1(\ell) = \prod_{p \mid \ell} \left(1 + \frac{1}{p - 1 - 1/p} \right) = \prod_{p \mid \ell} \left(1 + \frac{1}{\theta_1(p)} \right) = \sum_{d \mid \ell} \frac{\mu^2(d)}{\theta_1(d)}, \quad \text{say}. \tag{27}$$

Hence

$$J_5(x, Q) + J_6(x, Q) = C_3 \sum_{\ell < x/Q} \Bigg((x - \ell Q)(x \log x - x) \tag{28}$$

$$- \left(\frac{1}{2} x^2 \log x - \frac{3}{4} x^2 \right) + \frac{1}{2} (x - \ell Q)^2 \log (x - \ell Q) - \frac{3}{4} (x - \ell Q)^2 \Bigg) \frac{\psi_1(\ell)}{\ell} + O\left(\frac{x^2}{\log^A x} \right).$$

To complete the estimation of $J_5(x, Q) + J_6(x, Q)$ we need

Lemma 2. *Let*

$$C_4 = \prod_p \left(1 + \frac{1}{\theta_1(p) p} \right) \tag{29}$$

and let $B_1, B_2, \ldots$ denote specific constants whose actual values are not of importance here. Then, for $\xi \geqq 1$,

(i) $$\sum_{\ell < \xi} \frac{\psi_1(\ell)}{\ell} = C_4 \log \xi + B_1 + O\left(\frac{\log(\xi + 2)}{\xi} \right);$$

(ii) $$\sum_{\ell < \xi} (\xi - \ell) \frac{\psi_1(\ell)}{\ell} = C_4 \xi \log \xi + B_2 \xi + O\big(\log^2(\xi + 2) \big);$$

(iii) $$\sum_{\ell < \xi} (\xi - \ell)^2 \frac{\psi_1(\ell)}{\ell} = C_4 \xi^2 \log \xi + B_3 \xi^2 + O\big(\xi \log^2(\xi + 2) \big);$$

(iv) $$\sum_{\ell<\xi}\left\{\frac{1}{2}(\xi-\ell)^2\log(\xi-\ell)-\frac{3}{4}(\xi-\ell)^2\right\}\frac{\psi_1(\ell)}{\ell}$$
$$=\frac{1}{2}C_4\,\xi^2\log^2\xi+B_4\,\xi^2\log\xi+B_5\,\xi^2+O\big(\xi\log^3(\xi+2)\big),$$

where
$$B_4=\frac{1}{2}B_3-\frac{3}{4}C_4\,.$$

Parts (ii) and (iii) are obtained from single or double integrations of part (i), whose proof may be omitted because it is similar to that of such familiar formulae as the one occurring in part (i) of I, Lemma 1 (from which our (9) was derived); at this point it is helpful for our calculations to note that all these formulae are also trivially valid for $0\leqq\xi<1$ with an obvious interpretation for values of terms at $\xi=0$.

Going over to part (iv), we see that its left-hand side equals
$$\int_0^\xi\log(\xi-t)\sum_{\ell<t}(t-\ell)\frac{\psi_1(\ell)}{\ell}\,dt$$
because the latter is
$$\sum_{\ell<\xi}\frac{\psi_1(\ell)}{\ell}\int_\ell^\xi(t-\ell)\log(\xi-t)\,dt$$
$$=\sum_{\ell<\xi}\frac{\psi_1(\ell)}{\ell}\int_0^{\xi-\ell}(\xi-\ell-u)\log u\,du$$
in which the integral is the double integral $\frac{1}{2}u^2\log u-\frac{3}{4}u^2$ at $u=\xi-\ell$. Therefore, by part (ii), the sum to be estimated is
$$\int_0^\xi\log(\xi-t)(C_4\,t\log t+B_2\,t)\,dt+O\big(\xi\log^3(\xi+2)\big)$$
$$=C_4\int_0^\xi\log(\xi-t)\,t\log t\,dt+B_2\int_0^\xi\log(\xi-t)\,t\,dt+O\big(\xi\log^3(\xi+2)\big),$$
the first integral in the last line above being, via the substitution $t=\xi t'$,
$$\xi^2\int_0^1\{\log\xi+\log(1-t')\}\,t'(\log\xi+\log t')\,dt'$$
$$=\xi^2\log^2\xi\int_0^1 t'dt'+\xi^2\log\xi\left(\int_0^1 t'\log t'dt'+\int_0^1 t'\log(1-t')\,dt'\right)+B_6\,\xi^2$$
$$=\frac{1}{2}\xi^2\log^2\xi+\xi^2\log\xi\int_0^1\log t'dt'+B_6\,\xi^2=\frac{1}{2}\xi^2\log^2\xi-\xi^2\log\xi+B_6\,\xi^2$$
and the second one being

$$\xi^2 \log \xi \int_0^1 t' dt' + \xi^2 \int_0^1 t' \log(1-t')\, dt' = \frac{1}{2}\xi^2 \log \xi + B_7 \xi^2 .$$

Thus, altogether, we get the right side of (iv) with the value $B_4 = \frac{1}{2} B_2 - C_4$. Since the integration of (ii) to give (iii) implies that $B_3 = B_2 - \frac{1}{2} C_4$, the proof of the lemma is complete.

By (20) and (28) followed by Lemma 2,

$$\begin{aligned}
\frac{1}{C_3}\{S_5^*(x;Q_1,Q) + S_6^*(x;Q_1,Q)\} &= (x\log x - x)\Bigg(Q_1 \sum_{\ell < x/Q_1}\left(\frac{x}{Q_1} - \ell\right)\frac{\psi_1(\ell)}{\ell} \\
&\quad - Q \sum_{\ell < x/Q}\left(\frac{x}{Q} - \ell\right)\frac{\psi_1(\ell)}{\ell}\Bigg) - \left(\frac{1}{2}x^2\log x - \frac{3}{4}x^2\right)\sum_{x/Q < \ell \leqq x/Q_1}\frac{\psi_1(\ell)}{\ell} \\
&\quad + \frac{1}{2}\left(Q_1^2 \log Q_1 \sum_{\ell < x/Q_1}\left(\frac{x}{Q_1} - \ell\right)^2 \frac{\psi_1(\ell)}{\ell} - Q^2 \log Q \sum_{\ell < x/Q}\left(\frac{x}{Q} - \ell\right)^2 \frac{\psi_1(\ell)}{\ell}\right) \\
&\quad + Q_1^2 \sum_{\ell < x/Q_1}\left\{\frac{1}{2}\left(\frac{x}{Q_1} - \ell\right)^2 \log\left(\frac{x}{Q_1} - \ell\right) - \frac{3}{4}\left(\frac{x}{Q_1} - \ell\right)^2\right\}\frac{\psi_1(\ell)}{\ell} \\
&\quad - Q^2 \sum_{\ell < x/Q}\left\{\frac{1}{2}\left(\frac{x}{Q} - \ell\right)^2 \log\left(\frac{x}{Q} - \ell\right) - \frac{3}{4}\left(\frac{x}{Q} - \ell\right)^2\right\}\frac{\psi_1(\ell)}{\ell} + O\left(\frac{x^2}{\log^A x}\right) \\
&= C_4(x\log x - x)x\log\frac{Q}{Q_1} - C_4\left(\frac{1}{2}x^2\log x - \frac{3}{4}x^2\right)\log\frac{Q}{Q_1} + \frac{1}{2}C_4 x^2 \log Q_1 \log\frac{x}{Q_1} \\
&\quad - \frac{1}{2}C_4 x^2 \log Q \log\frac{x}{Q} - \frac{1}{2}B_3 x^2 \log\frac{Q}{Q_1} + \frac{1}{2}C_4 x^2\left(\log^2\frac{x}{Q_1} - \log^2\frac{x}{Q}\right) \\
&\quad + B_4 x^2 \log\frac{Q}{Q_1} + O\left(xQ\log x \log^2\frac{x}{Q}\right) + O\left(\frac{x^2}{\log^A x}\right) \\
&= x^2 \log x\left(C_4 - \frac{1}{2}C_4 - \frac{1}{2}C_4 + C_4\right)\log\frac{Q}{Q_1} \\
&\quad + x^2\left(-C_4 + \frac{3}{4}C_4 - \frac{1}{2}B_3 + B_4\right)\log\frac{Q}{Q_1} + O\left(xQ\log x\log^2\frac{x}{Q}\right) + O\left(\frac{x^2}{\log^A x}\right) \\
&= C_4 x^2 \log x \log\frac{Q}{Q_1} - C_4 x^2 \log\frac{Q}{Q_1} + O\left(xQ\log x\log^2\frac{x}{Q}\right) + O\left(\frac{x^2}{\log^A x}\right),
\end{aligned}$$

where (3) and (5) must be borne in mind. Therefore, as

$$C_3 C_4 = \prod_p \left(1 - \frac{1}{p(p-1)}\right)\left(1 + \frac{1}{p^2 - p - 1}\right) = 1$$

by (26) and (29), we infer that

$$S_5^*(x;Q_1,Q)+S_6^*(x;Q_1,Q)=x^2\log x\log\frac{Q}{Q_1}-x^2\log\frac{Q}{Q_1}+O\left(xQ\log x\log^2\frac{x}{Q}\right)$$

$$+O\left(\frac{x^2}{\log^A x}\right) \tag{30}$$

and conclude the treatment of the sums of intermediate difficulty.

5. Estimation of $S_4^*(x;Q_1,Q)$ – the preliminary stages and the application of the circle method

Having gained the foothills, we commence the main ascent by considering the sum $S_4^*(x;Q_1,Q)$ that is connected with (13) and (14). We confirm that (19) is still to hold and first find from (18), (13), and (14) that

$$J_4(x,Q)=\sum_{d\leqq x}\frac{\mu(d)}{d}\sum_{\substack{Q<k\leqq x\\ k\equiv 0,\,\mathrm{mod}\,d}}\ \sum_{\substack{p_2<p_3<p_1\leqq x\\ p_1\equiv p_2\equiv p_3,\,\mathrm{mod}\,k}}\log p_1\log p_2\log p_3 \tag{31}$$

$$=\sum_{d\leqq x}\frac{\mu(d)}{d}J^\dagger(x,Q;d),\quad\text{say}.$$

Next the congruential condition attached to the inner sum in $J^\dagger(x,Q;d)$ is there equivalent to the existence of *positive integers* ℓ_1, ℓ_2, and ℓ_3 such that

$$p_3-p_2=\ell_1k,\quad p_1-p_3=\ell_2k,\quad p_1-p_2=\ell_3k, \tag{32}$$

and

$$\ell_3=\ell_1+\ell_2, \tag{33}$$

where any pair of conditions in (32) is a substitute for the whole triplet when (33) holds. Also, if the common value of (ℓ_1,ℓ_2) and (ℓ_1,ℓ_2,ℓ_3) be δ with the consequence that we may write $\ell_1=\ell_1'\delta$, $\ell_2=\ell_2'\delta$, $\ell=\ell_3'\delta$ where

$$(\ell_1',\ell_2')=(\ell_1',\ell_2',\ell_3')=1\quad\text{and}\quad\ell_3'=\ell_1'+\ell_2', \tag{34}$$

the first two equations (for example) in (32) are tantamount to the pair

$$p_1\equiv p_2\equiv p_3,\,\mathrm{mod}\,\delta, \tag{35}$$

and

$$\ell_1'\{(p_1-p_3)/\delta\}=\ell_2'\{(p_3-p_2)/\delta\},$$

the latter member of which may be rewritten as

$$\ell_1'p_1+\ell_2'p_2-\ell_3'p_3=0. \tag{36}$$

Hence, taking into account the conditions on k that imply that

$$\ell_3 < x/Q \tag{37}$$

and that the congruence (35) holds, mod $d\delta$, we complete the first phase in the treatment of $J_4(x,Q)$ by deducing that

$$J^\dagger(x,Q;d) = \sum_{\delta < x/Q} \sum_{\ell_3' < x/Q\delta} \sum_{\substack{\ell_3' = \ell_1' + \ell_2' \\ (\ell_1', \ell_2') = 1}} P(x, Q\delta\ell_3'; \ell_1', \ell_2'; d\delta), \tag{38}$$

the innermost summand in which is defined by letting $\Theta = \Theta_{\Delta, \ell_1', \ell_2'}$ indicate the conjunction of the conditions (36) and $p_1 \equiv p_2 \equiv p_3$, mod Δ, and $(x\log^{-A_1} x < T < x)$ then setting

$$P(x,T;\ell_1',\ell_2';\Delta) = \sum_{\substack{\Theta \\ p_2 + T < p_1 \leqq x}} \log p_1 \log p_2 \log p_3\,. \tag{39}$$

The formula needed for $P(x,T;\ell_1',\ell_2';\Delta)$ is obtained by an appropriate variation of some version of the circle method that establishes Vinogradov's theorem on the representation of large odd numbers as the sum of three primes. Since no new principles are involved, we deem it sufficient to describe the main steps in the demonstration, particularly as the exhibition of all the extra details would become very wearisome. Adapted to facilitate comparison where possible with recent treatments of Vinogradov's theorem such as that given by Vaughan [4], our procedure seems the simplest available within the constraints imposed even though there is a small penalty to be paid in the shape of a minor Tauberian process at the end. Possibly, however, another programme would be preferable if one were to tackle the problem *ab initio* with the intention of producing a fully detailed proof.

First, as in certain preceding situations, the stipulation that $(p_1p_2p_3, \Delta) = 1$ may be included in the conditions in Θ without altering their effect. Secondly, in order to avoid a situation in which the generating function in the circle method is not a simple product of three independent sums over primes, we work initially with sums

$$P_1(x, t_1, t_2; \ell_1', \ell_2'; \Delta) = \sum_{\substack{\Theta \\ t_1 < p_1 \leqq x;\, p_2 \leqq t_2}} \log p_1 \log p_2 \log p_3 \tag{40}$$

that involve non-negative parameters t_1, t_2 such that $t_2 < t_1 \leqq x$, dissecting them into sums of segments P_2 through the decomposition expressed by

$$\begin{aligned} P_1(x, t_1, t_2; \ell_1', \ell_2'; \Delta) &= \sum_{\substack{0 < b \leqq \Delta \\ (b,\Delta) = 1}} \sum_{p_1 \equiv p_2 \equiv p_3 \equiv b,\ \mathrm{mod}\ \Delta} \\ &= \sum_{\substack{0 < b \leqq \Delta \\ (b,\Delta) = 1}} P_2(x, t_1, t_2; \ell_1', \ell_2'; b, \Delta), \quad \text{say}\,. \end{aligned} \tag{41}$$

Now ready for the introduction of the circle method, we form the three functions

$$f_j(\theta) = \sum_{p_j \equiv b,\ \mathrm{mod}\ \Delta} \log p_j \, e^{2\pi i \ell_j' p_j \theta} \quad (j = 1, 2)$$

and

$$f_3(\theta) = \sum_{p_3 \equiv b, \bmod \Delta} \log p_3 \, e^{-2\pi i \ell_3' p_3 \theta} \tag{42}$$

that produce the representation

$$P_2 = \int_0^1 f_1(\theta) f_2(\theta) f_3(\theta) \, d\theta \tag{43}$$

when the sums in (42) are subject to the conditions

$$t_1 < p_1 \leqq x; \quad p_2 \leqq t_2; \quad p_3 \leqq x \, . \tag{44}$$

It is therefore requisite to develop the appropriate properties of the generic sum

$$f(\theta) = \sum_{\substack{u < p \leqq v \\ p \equiv b, \bmod \Delta}} \log p \, e^{2\pi i \ell p \theta} \quad (0 \leqq u < v \leqq x)$$

of which $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ are particular examples.

Assuming throughout in conformity with (19) and (37) that $\ell \leqq \log^{A_1} x$ and then temporarily that $\Delta \leqq \log^{A_2} x$ for a sufficiently large constant $A_2 = A_2(A_1)$, we use a dissection of the range of integration of order $M = x \log^{-A_3} x$ in which the (non-intersecting) major arcs are of the form $|\theta - h/k| \leqq 1/M$ for rationals h/k in lowest terms with $k \leqq \log^{A_3} x$ and in which the residual set $\mathfrak{m}$ of θ is contained in the set of minor arcs given by

$$\left| \theta - \frac{h}{k} \right| \leqq \frac{1}{Mk} \quad \text{for} \quad \log^{A_3} x < k \leqq M \, . \tag{45}$$

On $\mathfrak{m}$ we express $f(\theta)$ as

$$\begin{aligned} &\frac{1}{\Delta} \sum_{u < p \leqq v} \log p \, e^{2\pi i \ell \theta p} \sum_{0 < c \leqq \Delta} e^{2\pi i c (p - b)/\Delta} \\ &= \frac{1}{\Delta} \sum_{0 < c \leqq \Delta} e^{-2\pi i b c/\Delta} \sum_{u < p \leqq v} \log p \, e^{2\pi i (\ell \theta + c/\Delta) p} \end{aligned} \tag{46}$$

and then, setting $M_1 = 2 \Delta M$, use Dirichlet's theorem to find a fraction h_1/k_1 in lowest terms (depending, in particular, on c) such that

$$\left| \ell \theta + \frac{c}{\Delta} - \frac{h_1}{k_1} \right| \leqq \frac{1}{M_1 k_1} \quad \text{and} \quad k_1 \leqq M_1 \, .$$

Since a basic technique in the practice of Diophantine approximation easily shews that this and (45) imply that $k_1 > k/2\Delta\ell$ and hence that

$$\frac{1}{2} \log^{A_3 - A_1 - A_2} x < k_1 \leqq 2x \log^{A_2 - A_3} x \, ,$$

a slight variation in the proof of Theorem 3.1 in [4] yields the estimate

$$O\left\{\log^4 x\left(xk_1^{-\frac{1}{2}}+x^{\frac{4}{5}}+x^{\frac{1}{2}}k_1^{\frac{1}{2}}\right)\right\}=O(x\log^{-A_4}x)$$

for the right-hand inner sum in (46) proved that A_3 be sufficiently large. Therefore $f(\theta)$ and thus any one of the $f_i(\theta)$ are subject to the same bound, and we thus conclude that the contribution of $\mathscr{m}$ to the integral in (43) is

$$O\left\{x\log^{-A_4}x\left(\int_0^1|f_1(\theta)|^2\,d\theta\right)^{\frac{1}{2}}\left(\int_0^1|f_2(\theta)|^2\right)^{\frac{1}{2}}\right\} \tag{47}$$

$$=O\left(x\log^{-A_4}x\sum_{p\leqq x}\log^2 p\right)=O(x^2\log^{-A_5}x)\,.$$

On each major arc centred on h/k we write $\theta=h/k+\phi$ and estimate $f(\theta)$ by considering the special value

$$f_4\left(\frac{h}{k}\right)=f\left(\frac{h}{k},w\right)=\sum_{\substack{u<p\leqq w\\ p\equiv b,\,\mathrm{mod}\,\Delta}}e^{2\pi i h\ell' p/k'}\log p\quad (u<w\leqq v)\,,$$

where $k'=k/(\ell,k)$, $\ell'=\ell/(\ell,k)$, and where therefore $(h\ell',k')=1$. By the prime number theorem for arithmetical progressions, for any positive constant A_6 we have

$$f_4\left(\frac{h}{k}\right)=\sum_{\substack{0<a'\leqq k'\\ (a',k')=1}}e^{2\pi i h\ell' a'/k'}\sum_{\substack{u<p\leqq w\\ p\equiv a',\,\mathrm{mod}\,k'\\ p\equiv b,\,\mathrm{mod}\,\Delta}}\log p+O\left(\sum_{p\mid k}\log p\right) \tag{48}$$

$$=\sum_{\substack{0<a'\leqq k'\\ (a',k')=1;\,(k',\Delta)\mid(a'-b)}}e^{2\pi i h\ell' a'/k'}\left\{\frac{w-u}{\phi([k',\Delta])}+O\left(\frac{x}{\log^{A_6}x}\right)\right\}+O(\log x)$$

$$=\frac{w-u}{\phi([k',\Delta])}\sum_{\substack{0<a'\leqq k'\\ (a',k')=1;\,a'\equiv b,\,\mathrm{mod}\,(k',\Delta)}}e^{2\pi i h\ell' a'/k'}+O\left(\frac{x}{\log^{A_6-A_3}x}\right)$$

on the assumption that all parameters occurring are subject to conditions already laid down either explicitly or implicitly. Next, if

$$k'=\prod_{p^\alpha\|k'}p^\alpha\,,$$

then let

$$k''=\prod_{p^\alpha\|k'';\,p\nmid\Delta}p^\alpha\,,$$

and deduce that the exponential sum in (48) is

$$\sum_{d\mid k''}\mu(d)\sum_{\substack{0<a'\leqq k'\\ a'\equiv 0,\,\mathrm{mod}\,d\\ a'\equiv b,\,\mathrm{mod}\,(k',\Delta)}}e^{2\pi i h\ell' a'/k'}\,,$$

the inner sum in which is taken over an arithmetical progression whose common difference $d(k',\Delta)$ is certainly a proper division of k' unless $d = k''$ and $\big(k'/(k',\Delta),\Delta\big) = 1$. Therefore this exponential sum is zero save when $\big(k'/(k',\Delta),\Delta\big) = 1$, in which case

$$f_4\left(\frac{h}{k}\right) = \frac{\mu\{k'/(k',\Delta)\}}{\phi\{[k',\Delta]\}}(w-u)\,e^{2\pi i h\ell' a'/k'} + O\left(\frac{x}{\log^{A_7} x}\right)$$

where now a' is the unique root, mod k', of the simultaneous congruences

$$v \equiv 0, \bmod k'/(k',\Delta), \quad v \equiv b, \bmod (k',\Delta)\,. \tag{49}$$

If, however, $\big(k'/(k',\Delta),\Delta\big) > 1$, then

$$f_4\left(\frac{h}{k}\right) = O\left(\frac{x}{\log^{A_7} x}\right).$$

Passing on to $f(\theta) = f(h/k + \phi)$ through partial summation and the deployment of the function

$$v(\phi) = \int_u^v e^{2\pi i \ell z \phi}\,dz\,, \tag{50}$$

we then deduce in the usual way that

$$\begin{aligned} f(\theta) &= \frac{\mu\{k'/(k',\Delta)\}}{\phi\{[k',\Delta]\}}v(\phi)\,e^{2\pi i h\ell' a'/k'} + O\left(\frac{x}{\log^{A_7} x}\right) + O\left(\frac{x}{\log^{A_7} x}\int_0^x |\ell\phi|\,dz\right) \\ &= \frac{\mu\{k'/(k',\Delta)\}}{\phi\{[k',\Delta]\}}v(\phi)\,e^{2\pi i h\ell' a'/k'} + O\left(\frac{x}{\log^{A_7 - A_1 - A_3} x}\right) \\ &= \frac{\mu\{k'/(k',\Delta)\}}{\phi\{[k',\Delta]\}}v(\phi)\,e^{2\pi i h\ell' a'/k'} + O\left(\frac{x}{\log^{A_8} x}\right) \end{aligned}$$

if $\big(k'/(k',\Delta),\Delta\big) = 1$ but that

$$f(\theta) = O\left(\frac{x}{\log^{A_8} x}\right)$$

otherwise.

To apply these estimates to the evaluation of the integral in (43) over the major arcs, we specialize them for each function $f_j(\theta)$ and denote by k_j, a_j, and $v_j(\phi)$ the entities that correspond to k', a', and $v(\phi)$ in the above work. Then the condition $\big(k/(k,\Delta),\Delta\big) = 1$ is equivalent to the conjunction of $\big(k_j/(k_j,\Delta),\Delta\big) = 1$ for $i = 1, 2, 3$, the congruence (49) with k' replaced by k then supplying a simultaneous solution a of (49) for the values k_1, k_2, k_3 of k'. Hence in this instance

$$f_1(\theta)f_2(\theta)f_3(\theta) = v_1(\phi)v_2(\phi)v_3(\phi)e^{2\pi iha(\ell_1'+\ell_2'-\ell_3')/k}\prod_{1\leqq j\leqq 3}\frac{\mu\{k_j/(k_j,\Delta)\}}{\phi([k_j,\Delta])}+O\left(\frac{x^3}{\log^{A_8}x}\right)$$

$$= v_1(\phi)v_2(\phi)v_3(\phi)\prod_{1\leqq j\leqq 3}\frac{\mu\{k_j/(k_j,\Delta)\}}{\phi([k_j,\Delta])}+O\left(\frac{x^3}{\log^{A_8}x}\right)$$

whereas, in the contrary case,

$$f_1(\theta)f_2(\theta)f_3(\theta) = O\left(\frac{x^3}{\log^{A_8}x}\right).$$

Consequently, summing over all relatively prime h, k satisfying $0<h\leqq k\leqq M$ and then integrating with respect to ϕ over the interval $(-1/M, 1/M)$, we see via (50) that the contribution of the major arcs to P_2 is

$$(51)\quad \left(\sum_{\substack{k\leqq \log^{A_3}x\\ (k/(k,\Delta),\Delta)=1}}\phi(k)\prod_{1\leqq j\leqq 3}\frac{\mu\{k_j/(k_j,\Delta)\}}{\phi([k_j,\Delta])}\right)\int_{-1/M}^{1/M}v_1(\phi)v_2(\phi)v_3(\phi)\,d\phi + O\left(\frac{x^2}{\log^{A_8-3A_3}x}\right)$$

$$=\left\{\sum_{(k/(k,\Delta),\Delta)=1}\phi k)\prod_{1\leqq j\leqq 3}\frac{\mu\{k_j/(k_j,\Delta)\}}{\phi([k_j,\Delta])}+O\left(\frac{1}{\log^{A_3}x}\right)\right\}\left\{\int_{-\infty}^{\infty}v_1(\phi)v_2(\phi)v_3(\phi)\,d\phi + O\left(\frac{1}{\ell_1'\ell_2'\ell_3'}\int_{1/M}^{\infty}\frac{d\phi}{\phi^3}\right)\right\}+O\left(\frac{x^2}{\log^{A_8-3A_3}x}\right)$$

$$=\left\{\mathfrak{S}_{\Delta,\ell_1',\ell_2',\ell_3'}+O\left(\frac{1}{\log^{A_3}x}\right)\right\}I_{t_1,t_2,x}+O\left(\frac{x^2}{\log^{A_9}x}\right),\quad\text{say},$$

provided that A_3 and $A_6=A_6(A_3)$ be sufficiently large.

The integral I is evaluated in the standard way by Fourier's integral theorem[1]). Arising first as a triple integral with variables of integration z_1, z_2, z_3, the integrand is transformed by means of the substitution

$$Z_1=z_1,\quad Z_2=z_2,\quad Z=\ell_1'z_1+\ell_2'z_2-\ell_3'z_3,$$

of absolute modulus ℓ_3 so it adopts the guise of the Fourier transform

$$\frac{1}{\ell_3'}\int_{-\infty}^{\infty}F(Z)e^{2\pi iZ\phi}\,dZ.$$

---

[1]) It seems we must part company with Vaughan's treatment at this point because his application of the sum $\nu(\beta)$ in equation (3.6) of [4] seems only to be appropriate when the coefficients of all three primes in an equation of type $m_1p_1+m_2p_2+m_3p_3=n$ are equal to ± 1.

Consequently

$$I = \frac{1}{\ell_3'} F(0) = \frac{(x - t_1)\, t_2}{\ell_3'} \tag{52}$$

because the limits for z_1, z_2 implied by the first two constituents of (44) imply that $z_3 \leqq x$ when $\ell_1' z_1 + \ell_2' z_2 - \ell_3' z_3 = 0$ and $\ell_1' + \ell_2' = \ell_3'$.

To evaluate the singular series by Euler's theorem, first note that

$$\phi([k_j, \Delta]) = \phi\{\Delta (k_j/(k_j, \Delta))\} = \phi(\Delta)\, \phi\, \{k_j\, /(k_j, \Delta)\}$$

when $(k/(k, \Delta), \Delta) = 1$ so that the general term in the series is

$$\frac{1}{\phi^3(\Delta)}\, \phi(k) \prod_{1 \leqq j \leqq 3} \frac{\mu\{k_j/(k_j, \Delta)\}}{\phi\{k_j/(k_j, \Delta)\}} \quad ((k/(k, \Delta), \Delta) = 1),$$

in which the multiplier of $1/\phi^3(\Delta)$ is multiplicative in k. Therefore, remembering that $\ell_1', \ell_2', \ell_3'$ are co-prime in pairs and that one of them is even, we have

$$\begin{aligned}
\mathfrak{S} &= \frac{1}{\phi^3(\Delta)} \prod_{p \nmid \ell_1'\ell_2'\ell_3'\Delta} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid \ell_1'\ell_2'\ell_3' \\ p \nmid \Delta}} \left(1 + \frac{1}{(p-1)}\right) \prod_{p^\beta \| \Delta} (1 + \phi(p) + \cdots + \phi(p^\beta)) \\
&= \frac{\Delta}{\phi^3(\Delta)} \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid \ell_1'\ell_2'\ell_3' \\ p \nmid 2\Delta}} \left(1 - \frac{1}{(p-1)}\right)^{-1} \\
&\quad \times \prod_{\substack{p \mid \Delta \\ p > 2}} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \prod_{\substack{p = 2 \\ p \nmid \Delta}} \left(1 + \frac{1}{(p-1)}\right) \\
&= \frac{1}{\Delta \phi(\Delta)} \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid \ell_1'\ell_2'\ell_3' \\ p \nmid 2\Delta}} \left(1 - \frac{1}{(p-1)}\right)^{-1} \prod_{\substack{p \mid \Delta \\ p > 2}} \left(1 - \frac{2}{p}\right)^{-1} \prod_{\substack{p \mid \Delta \\ p = 2}} 4 \prod_{\substack{p \nmid \Delta \\ p = 2}} 2 \\
&= \frac{2 C_5}{\Delta \phi(\Delta)} E(\Delta) F(\Delta) G_\Delta(\ell_1'\ell_2'\ell_3'),
\end{aligned} \tag{53}$$

where

$$\left\{ \begin{aligned}
&E(\Delta) = \begin{cases} 2, & \text{if } \Delta \text{ be even}, \\ 1, & \text{if } \Delta \text{ be odd}, \end{cases} \\
&C_5 = \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right), \quad F(\Delta) = \prod_{p \mid \Delta;\, p > 2} \left(1 + \frac{2}{p - 2}\right), \\
&\text{and} \quad G_\Delta(L) = \prod_{p \mid L;\, p \nmid 2\Delta} \left(1 + \frac{1}{p - 2}\right).
\end{aligned} \right. \tag{54}$$

The summation of the content of equations (47), (51), (52), and (53) produces an asymptotic formula for P_2, which when summed over b as in (41) yields

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$$P_1(x, t_1, t_2; \ell_1', \ell_2'; \Delta) = \frac{2C_5}{\Delta \ell_3'}(x - t_1)\, t_2\, E(\Delta)\, F(\Delta)\, G_\Delta(\ell_1'\ell_2'\ell_3') + O\left(\frac{x}{\log^{A_{10}} x}\right).$$

From this, by (39) and (40) and some simple calculations involving interchanges in the orders of summation and integration, it follows that

$$\begin{aligned} P_3(T) = \int_T^x P(x, T_1; \ell_1', \ell_2'; \Delta)\, dT_1 &= \sum_{\substack{\Theta \\ p_2 + T < p_1 \leqq x}} (p_1 - p_2 - T) \log p_1 \log p_2 \log p_3 \\ &= \sum_{\substack{\Theta \\ p_2 + T < p_1 \leqq x}} \log p_1 \log p_2 \log p_3 \int_{p_2}^{p_1 - T} dt \\ &= \int_0^{x-T} P_1(x, T + t, t; \ell_1', \ell_2'; \Delta)\, dt \\ &= \frac{2C_5}{\Delta \ell_3'} E(\Delta) F(\Delta) G_\Delta(\ell_1'\ell_2'\ell_3') \int_0^{x-T} (x - T - t)\, t\, dt + O\left(\frac{x^2(x-T)}{\log^{A_{10}} x}\right) \\ &= \frac{C_5}{3\Delta \ell_3'} E(\Delta) F(\Delta) G_\Delta(\ell_1'\ell_2'\ell_3')(x - T)^3 + O\left(\frac{x^2(x-T)}{\log^{A_{10}} x}\right), \end{aligned}$$

which is an Abelian version of the result we seek. To extract what is needed, we perform a "de la Vallée Poussin differentiation" by choosing H such that $0 < H < x - T$, T and using the inequality

$$\frac{1}{H}\left\{P_3(T) - P_3(T + H)\right\} \leqq P(x, T) \leqq \frac{1}{H}\left\{P_3(T - H) - P_3(T)\right\}$$

that implies that

$$P(x, T) = \frac{C_5}{\Delta \ell_3'} E(\Delta) F(\Delta) G_\Delta(\ell_1'\ell_2'\ell_3')(x - T)^2 + O\left\{H(x - T)^2\right\} + O\left(\frac{x^2(x-T)}{H \log^{A_{10}} x}\right)$$

in view of well-known inequalities for divisor-type functions. Hence, setting

$$H = (x - T) \log^{-\frac{1}{2}A_{10}} x$$

and confirming through (39) that $H < x \log^{-A_1} x < T$ for sufficiently large A_1, we conclude that

$$P(x, T; \ell_1', \ell_2'; \Delta) = \frac{C_5}{\Delta \ell_3'} E(\Delta) F(\Delta) G_\Delta(\ell_1'\ell_2'\ell_3')(x - T)^2 + O\left(\frac{x^2}{\log^{A_{11}} x}\right) \tag{55}$$

when $\Delta, \ell_1', \ell_2', \ell_3'$ satisfy the stipulations laid down at the point where the dissection of the unit circle was introduced. But, since $P(x)$ certainly does not exceed $\log^3 x$ times the number of positive integers n_1, n_2 not exceeding x that are congruent to each other, mod Δ, we have

$$P(x,T) - \frac{C_5}{\Delta \ell_3'} E(\Delta) F(\Delta) G_\Delta(\ell_1'\ell_2'\ell_3')(x-T)^2$$
$$= O\left\{\Delta \log^3 x \left(\frac{x}{\Delta} + 0(1)\right)^2\right\} + O\left(\frac{x^2 \log x}{\Delta}\right)$$
$$= O\left(\frac{x^2 \log^3 x}{\Delta}\right)$$
$$= O\left(\frac{x^2}{\log^{A_{11}} x}\right)$$

for $\log^{A_2} x < \Delta \leqq x$ and A_2 sufficiently large; also, this result is trivial for $\Delta > x$ because then $P(x,T)$ is zero from its genesis at the beginning of this section. Thus (55) is valid for all values of ℓ_1', ℓ_2' in question whatever the value of Δ.

Having secured our formula for $P(x,T)$, we prepare for its use by estimating

$$(56) \qquad P_3(x, Q\delta\ell_3'; \ell_1', \ell_2'; \delta) = \sum_{d \leqq x} \frac{\mu(d)}{d} P(x, Q\delta\ell_3'; \ell_1', \ell_2'; d\delta),$$

which is the innermost sum in the quadruple sum obtained by substituting the right side of (55) in (31) by way of (38) and then first summing over d. We get

$$(57) \qquad P_3(x, Q\delta\ell_3'; \ell_1', \ell_2'; \delta)$$
$$= \frac{C_5}{\delta\ell_3'}(x - Q\delta\ell_3')^2 \sum_{d \leqq x} \frac{\mu(d) E(d\delta) F(d\delta) G_{d\delta}(\ell_1'\ell_2'\ell_3')}{d^2} + O\left(\frac{x^2}{\log^{A_{11}} x} \sum_{d \leqq x} \frac{1}{d}\right)$$
$$= \frac{C_5}{\delta\ell_3'}(x - Q\delta\ell_3')^2 \sum_{d=1}^{\infty} \frac{\mu(d) E(d\delta) F(d\delta) G_{d\delta}(\ell_1'\ell_2'\ell_3')}{d^2} + O\left(x^2 \sum_{d > x} \frac{1}{d^2}\right) + O\left(\frac{x^2}{\log^{A_{12}} x}\right)$$
$$= \frac{C_5}{\delta\ell_3'}(x - Q\delta\ell_3')^2 B(\delta, \ell_1'\ell_2'\ell_3') + O\left(\frac{x^2}{\log^{A_{12}} x}\right), \quad \text{say},$$

the factor $B(\delta, \ell_1'\ell_2'\ell_3')$ being evaluated by the relations

$$F(d\delta) = F(\delta) \prod_{\substack{p \mid d;\, p \nmid \delta \\ p > 2}} \left(1 + \frac{2}{p-2}\right) = F(\delta) F_\delta(d), \quad \text{say},$$

$$G_{d\delta}(L) = G_\delta(L) \prod_{\substack{p \mid L \\ p \nmid 2\delta;\, p \mid d}} \left(1 - \frac{1}{p-1}\right) = G_\delta(L) G_{\delta,d}(L), \quad \text{say},$$

and

$$E(d\delta) = \begin{cases} 2, & \text{if } \delta \text{ even}, \\ E(d), & \text{if } \delta \text{ odd}, \end{cases}$$

that flow from (54). Whereupon, for δ even, we have

$$B(\delta, L) = 2F(\delta)\,G_\delta(L) \sum_{d=1}^{\infty} \frac{\mu(d)\,F_\delta(d)\,G_{\delta,d}(L)}{d^2}$$

$$= 2F(\delta)\,G_\delta(L) \prod_{p\mid\delta}\left(1-\frac{1}{p^2}\right) \prod_{\substack{p\nmid\delta\\ p\mid L}}\left(1-\frac{1}{p(p-1)}\right) \prod_{\substack{p\nmid\delta\\ p\nmid L}}\left(1-\frac{1}{p(p-2)}\right)$$

$$= \frac{3}{2}F(\delta)\,G_\delta(L) \prod_{p>2}\left(1-\frac{1}{p(p-2)}\right) \prod_{\substack{p\mid\delta\\ p>2}}\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p(p-2)}\right)^{-1}$$

$$\times \prod_{p\mid L;\, p\nmid\delta}\left(1-\frac{1}{p(p-1)}\right)\left(1-\frac{1}{p(p-2)}\right)^{-1}$$

$$= \frac{3}{2}\prod_{p>2}\left(1-\frac{1}{p(p-2)}\right) \prod_{\substack{p\mid\delta\\ p>2}}\left(1+\frac{2}{p-2-1/p}\right) \prod_{p\mid L;\, p\nmid\delta}\left(1+\frac{1}{p-2-1/p}\right)$$

by Euler's theorem and (54). A similar calculation being applicable when δ is odd, we conclude that

$$B(\delta, L) = \frac{1}{2}C_6\,E^*(\delta)\,\Phi(\delta)\,\Gamma_\delta(L)\,, \tag{58}$$

in which, setting

$$\theta_2(p) = p - 2 - \frac{1}{p} \quad (p \text{ odd}), \quad \theta_2(n) = \prod_{\substack{p\mid n\\ p>2}} \theta_2(p)\,, \tag{59}$$

(compare with the definition of $\theta_1(n)$ in (27)), we have

$$\left\{\begin{aligned} &C_6 = \prod_{p>2}\left(1-\frac{1}{p(p-2)}\right), \quad \Phi(\delta) = \prod_{p\mid\delta;\, p>2}\left(1+\frac{2}{\theta_2(p)}\right), \\ &\Gamma_\delta(L) = \prod_{p\mid L;\, p\nmid 2\delta}\left(1+\frac{1}{\theta_2(p)}\right) = \sum_{\substack{d\mid L\\ (d,2\delta)=1}} \frac{\mu^2(d)}{\theta_2(d)}, \\ &\text{and} \qquad E^*(\delta) = \begin{cases} 3, & \text{if } \delta \text{ even}, \\ 1, & \text{if } \delta \text{ odd}. \end{cases} \end{aligned}\right. \tag{60}$$

Our initial preparations for the estimation of $S_4^*(x; Q_1, Q)$ are complete save for a final transformation that depends in part on the relations

$$\Gamma_{\delta_1}(d\delta_2) = \Gamma_{\delta_1}(d)\,\Gamma_{d\delta_1}(\delta_2), \quad \Gamma_d(\delta) = \Gamma(d\delta)/\Gamma(d) \tag{61}$$

where

$$\Gamma(d) = \Gamma_1(d)\,. \tag{62}$$

To introduce this step let us gather up what has been achieved in (57) and (58) to infer that

$$J_4(x;Q) = \frac{1}{2} C_5 C_6 \sum_{\delta < x/Q} \frac{E^*(\delta)\Phi(\delta)}{\delta} \sum_{\ell' < x/Q\delta} \frac{(x - Q\delta\ell')^2}{\ell'} \Gamma_\delta(\ell') \sum_{\substack{\ell_1' + \ell_2' = \ell' \\ (\ell_1', \ell_2') = 1}} \Gamma_\delta(\ell_1')\Gamma_\delta(\ell_2')$$

$$+ O\left(\frac{x^2}{\log^{A_{11}} x} \sum_{\delta < x/Q} \sum_{\ell_1', \ell_2' < x/Q\delta} 1\right)$$

$$= \frac{1}{2} C_5 C_6 Q^2 \sum_{\delta < x/Q} E^*(\delta)\Phi(\delta)\delta \sum_{\ell' < x/Q\delta} \frac{(x/Q\delta - \ell')^2}{\ell'} \Gamma_\delta(\ell') \sum_{\substack{\ell_1' + \ell_2' = \ell' \\ (\ell_1', \ell_2') = 1}} \Gamma_\delta(\ell_1')\Gamma_\delta(\ell_2')$$

$$+ O\left(\frac{x^2}{\log^{A_{11} - 2A_2} x}\right)$$

$$(63) \qquad = \frac{1}{2} C_5 C_6 Q^2 \mathscr{J}(x/Q) + O\left(\frac{x^2}{\log^A x}\right), \quad \text{say},$$

in view of (19). The coprimality condition in the innermost sum being an impediment to the future treatment of $\mathscr{J}(y)$, we remove it by affecting the summand in it with the factor

$$\sum_{d \mid \ell_1'; d \mid \ell_2'} \mu(d),$$

whence, if we write $\ell_1' = \ell_1 d$, $\ell_2' = \ell_2 d$, and change the order of summation, we arrive at

$$\mathscr{J}(y) = \sum_{\delta < y} E^*(\delta)\Phi(\delta)\delta \sum_{d < y/\delta} \mu(d) \sum_{\ell < y/\delta d} \frac{(y/\delta - d\ell)^2}{d\ell} \Gamma_\delta(d\ell) \sum_{\ell_1 + \ell_2 = \ell} \Gamma_\delta(d\ell_1)\Gamma_\delta(d\ell_2)$$

$$= \sum_{\delta d < y} \mu(d) E^*(\delta)\Phi(\delta)\delta d \Gamma_\delta^3(d) \sum_{\ell < y/\delta d} \frac{(y/\delta d - \ell)^2}{\ell} \Gamma_{\delta d}(\ell) \sum_{\ell_1 + \ell_2 = \ell} \Gamma_{\delta d}(\ell_1)\Gamma_{\delta d}(\ell_2)$$

$$= \sum_{\Delta < y} \Delta I(\Delta) \sum_{\ell < y/\Delta} \frac{(y/\Delta - \ell)^2}{\ell} \Gamma_\Delta(\ell) \sum_{\ell_1 + \ell_2 = \ell} \Gamma_\Delta(\ell_1)\Gamma_\Delta(\ell_2)$$

$$(64) \qquad = \sum_{\Delta < y} \Delta I(\Delta) \mathscr{J}_1(y/\Delta, \Delta), \quad \text{say},$$

in virtue of (61).

Furthermore, by (61) and (62), we find that $I(\Delta)$ is the multiplicative function

$$\Gamma^3(\Delta) \sum_{d\delta = \Delta} \frac{\mu(d) E^*(\delta)\Phi(\delta)}{\Gamma^3(\delta)},$$

which for an odd prime power p^α is

$$\Phi(p) - \Gamma^3(p)$$

or zero according as $\alpha = 1$ or $\alpha > 1$; similarly for $\Delta = 2^\alpha$ it is

$$E^*(2) - 1 = 2$$

when $\alpha = 1$ but is zero otherwise. Hence, as

$$\Gamma^3(p) - \Phi(p) = \left(1 + \frac{1}{\theta_2(p)}\right)^3 - \left(1 + \frac{2}{\theta_2(p)}\right) = \frac{1}{\theta_2(p)}\left(1 + \frac{3}{\theta_2(p)} + \frac{1}{\theta_2^2(p)}\right)$$

$$= \frac{\tau(p)}{\theta_2(p)}, \quad \text{say},$$

for $p > 2$ by (60) and (62), we conclude from (64) that[2])

$$\mathscr{I}(y) = \sum_{\Delta \leqq y} \frac{\mu(\Delta)\Delta E^\dagger(\Delta)\tau(\Delta)}{\theta_2(\Delta)} \mathscr{I}_1(y/\Delta, \Delta) \tag{65}$$

on the understanding that

$$\tau(\Delta) = \prod_{p|\Delta;\, p>2} \tau(p), \tag{66}$$

and $E^\dagger(\Delta) = 1$ or -2 according as Δ is odd or even.

6. The programme for the analysis of $\mathscr{I}_1(z, \Delta)$

We have reached a point on our ascent where some nicety in the analysis is needed because several plausible paths we might take do not lead in the right direction. The source of the estimation of $\mathscr{I}_1(z, \Delta)$ is essentially the following corollary

$$\mathscr{I}_1(z, \Delta) = \sum_{\ell < z} \frac{(z-\ell)^2}{\ell} \sum_{\substack{d|\ell \\ (d, 2\Delta) = 1}} \frac{\mu^2(d)}{\theta_2(d)} \sum_{\ell_1 + \ell_2 = \ell} \Gamma_\Delta(\ell_1)\Gamma_\Delta(\ell_2) \tag{67}$$

$$= \sum_{\substack{d < z \\ (d, 2\Delta) = 1}} \frac{\mu^2(d)}{\theta_2(d)} \sum_{\substack{\ell < z \\ \ell \equiv 0, \bmod d}} \frac{(z-\ell)^2}{\ell} \sum_{\ell_1 + \ell_2 = \ell} \Gamma_\Delta(\ell_1)\Gamma_\Delta(\ell_2)$$

$$= \sum_{\substack{d < z \\ (d, 2\Delta) = 1}} \frac{\mu^2(d)\mathscr{I}_1^*(z, \Delta; d)}{\theta_2(d)}, \quad \text{say},$$

of (64) and (60) and the fact that there are two different methods for dealing with $\mathscr{I}_1^*(z, \Delta; d)$, one of which is unsatisfactory for larger values of d and the other for smaller values of d. Although there is an intermediate range of d for which both methods yield significant information, direct use of the two consequential estimates in the summations over d and Δ does not lead to adequately small remainder terms in the ensuing answer.

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[2]) Owing to the presence of factors such as $(m - v)^2$ in sums over a variable m, it is frequently immaterial whether the upper limit of summation is given by $m < v$ or $m \leqq v$. We choose the inequality that seems most natural in each context.

Instead, we must compare the two estimates for $\mathscr{J}_1^*(z,\Delta;d)$ in the common area of importance in order to fashion a new expression that permits the summation over d to be performed in a semi-implicit and satisfactory manner.

Although the programme could be directly applied as described to $\mathscr{J}_1(z,\Delta)$ itself, it actually proves better for technical reasons to aim it in the first place at the surrogate sum

$$\mathscr{J}_2(z_1,\Delta) = \sum_{\ell < z_1} (z_1-\ell)^2 \ell \Gamma_\Delta(\ell) \sum_{\ell_1+\ell_2=\ell} \Gamma_\Delta(\ell_1)\Gamma_\Delta(\ell_2) \tag{68}$$

$$= \sum_{\substack{d<z_1 \\ (d,2\Delta)=1}} \frac{\mu^2(d)}{\theta_2(d)} \sum_{\substack{\ell<z_1 \\ \ell\equiv 0, \bmod d}} (z_1-\ell)^2 \ell \sum_{\ell_1+\ell_2=\ell} \Gamma_\Delta(\ell_1)\Gamma_\Delta(\ell_2)$$

$$= \sum_{\substack{d<z_1 \\ (d,2\Delta)=1}} \frac{\mu^2(d)\mathscr{J}_2^*(z_1,\Delta;d)}{\theta_2(d)}, \quad \text{say},$$

which is connected with $\mathscr{J}_1(z,\Delta)$ by means of

Lemma 3. *Let*

$$s_0(u) = \sum_{n\leqq u} \frac{(u-n)^2 a_n}{n}, \quad s_2(u) = \sum_{n\leqq u} (u-n)^2 n a_n,$$

where n denotes a positive integer and $u>0$. Then

$$s_0(u) = \frac{s_2(u)}{u^2} - 6u\int_0^u \frac{s_2(t)\,dt}{t^4} + 12u^2 \int_0^u \frac{s_2(t)\,dt}{t^5}.$$

First, if

$$s_1(u) = \sum_{n\leqq u} (u-n)^2 a_n,$$

then

$$s_0(u) = \frac{s_1(u)}{u} + 3u^2 \int_0^u \frac{s_1(t)\,dt}{t^4}. \tag{69}$$

Not being a mere consequence of partial summation, this is verified by noting that

$$s_0(u) - \frac{s_1(u)}{u} = \frac{1}{u}\sum_{n\leqq u} \frac{(u-n)^3 a_n}{n}$$

and that

$$3u^2\int_0^u \frac{s_1(t)\,dt}{t^4} = 3u^2 \int_0^u \sum_{n\leqq t}(t-n)^2 a_n \frac{dt}{t^4} = 3u^2 \sum_{n\leqq u} a_n \int_n^u \frac{(t-n)^2\,dt}{t^4}$$

$$= \frac{1}{u}\sum_{n\leqq u}\frac{(u-n)^3 a_n}{n}$$

by an easy integration. Then, stating $s_1(u)$ in terms of $s_2(u)$ by means of the formula (69) for the series with na_n in place of a_n, we obtain the lemma by substituting the resulting expression for $s_1(u)$ in the same formula (69) and transforming into a single integral the double integral that arises.

Lastly, in anticipation of what is to come, it is helpful to make some comment on future notational conventions. Owing their origin in an increasingly indirect and implicit manner from the originally given x and Q, the entities y, z, z_i, Δ, d will where necessary be assumed to be subject to the restraints

$$(70)\quad y = x/Q;\quad \Delta \leqq y;\quad z = y/\Delta \geqq 1;\quad 1 \leqq z_1 \leqq z;\quad 0 < z_2 \leqq z_1;\quad d \leqq z;$$
$$d \text{ square-free};\quad (d, 2\Delta) = 1$$

and any other stipulations that are implicitly imposed by defined domains of summation.

7. Estimation of $\mathscr{J}_2^*(z_1, \Delta; d)$ – first method

If we scrutinize the conditions of summation in the sum $\mathscr{J}_2^*(z_1, \Delta; d)$ tacitly defined in (68), we see that (ℓ_1, d), (ℓ_2, d) in the inner addition have a common value e, say, dividing d, wherefore we set

$$\ell_1 = e\ell_1',\quad \ell_2 = e\ell_2',\quad d' = d/e,\quad \Delta' = \Delta e$$

so that

$$(71)\qquad \Delta d = \Delta' d',\quad (d', 2\Delta') = 1,\quad (\ell_1'\ell_2', d') = 1$$

because of (70). Hence, then suppressing the dashes from ℓ_1', ℓ_2' for notational convenience and exploiting (61), we have

$$\mathscr{J}_2^*(z_1, \Delta; d) = \sum_{e|d} e^3 \Gamma^2(e) \sum_{\substack{\ell_1+\ell_2 \leqq z_1/e \\ \ell_1+\ell_2 \equiv 0, \bmod d' \\ (\ell_1\ell_2, d')=1}} \left(\frac{z_1}{e} - \ell_1 - \ell_2\right)^2 (\ell_1+\ell_2)\Gamma_{\Delta'}(\ell_1)\Gamma_{\Delta'}(\ell_2)$$

$$(72)\qquad = \sum_{e|d} e^3 \Gamma_\Delta^2(e) \mathscr{J}_3(z_1/e, \Delta'; d'),\quad \text{say},$$

wherein

$$\mathscr{J}_3(z_2, \Delta'; d') = 2 \sum_{\substack{\ell_1+\ell_2 \leqq z_2 \\ \ell_1+\ell_2 \equiv 0, \bmod d' \\ (\ell_1\ell_2, d')=1}} (z_2 - \ell_1 - \ell_2)^2 \Gamma_{\Delta'}(\ell_1)\ell_2\Gamma_{\Delta'}(\ell_2)$$

by symmetry.

Even now we are not yet quite ready to commence detailed calculations, since we need to add one last link to the chain of transformations of $J_4(x; Q)$ through the theory of characters, denoting a character χ, mod c, by χ_c or χ_c^* according as it is principal or non-principal. We end up with the equation

$$\mathscr{J}_3(z_2, \Delta'; d') = \frac{2}{\phi(d')} \sum_{\ell_1 + \ell_2 = z_2} (z_2 - \ell_1 - \ell_2)^2 \Gamma_{\Delta'}(\ell_1) \ell_2 \Gamma_{\Delta'}(\ell_2) \sum_{\chi, \bmod d'} \chi(-1) \chi(\ell_1) \bar{\chi}(\ell_2)$$

$$= \frac{2}{\phi(d')} \sum_{\ell_1 + \ell_2 = z_2} (z_2 - \ell_1 - \ell_2)^2 \chi_{d'}(\ell_1) \Gamma_{\Delta'}(\ell_1) \chi_{d'}(\ell_2) \ell_2 \Gamma_{\Delta'}(\ell_2)$$

$$+ \frac{2}{\phi(d')} \sum_{\chi_{d'}^*} \chi_{d'}^*(-1) \sum_{\ell_1 + \ell_2 = z_2} (z_2 - \ell_1 - \ell_2)^2 \chi_{d'}^*(\ell_1) \Gamma_{\Delta'}(\ell_1) \bar{\chi}_{d'}^*(\ell_2) \ell_2 \Gamma_{\Delta'}(\ell_2)$$

$$= \frac{2}{\phi(d')} \mathscr{J}_4(z_2, \Delta'; d') + \frac{2}{\phi(d')} \sum_{\chi_{d'}^*} \chi_{d'}^*(-1) \mathscr{J}_5(z_2, \Delta'; \chi_{d'}^*)$$

$$(73) \qquad = \frac{2}{\phi(d')} \mathscr{J}_4(z_2, \Delta'; d') + \frac{2}{\phi(d')} \mathscr{J}_6(z_2, \Delta'; d'), \quad \text{say},$$

whose constituents are in a form suitable for the analysis we plan.

We consider $\mathscr{J}_4$ first, stating at once that its evaluation depends on whether $d' = 1$, or d' be a prime ϖ, or d' have more than one prime factor. Being obviously the generating function for the inner sum in the representation

$$(74) \qquad \sum_{\ell_1 < z_2} \chi_{d'}(\ell_1) \Gamma_{\Delta'}(\ell_1) \sum_{\ell_2 < (z_2 - \ell_1)} (z_2 - \ell_1 - \ell_2)^2 \chi_{d'}(\ell_2) \ell_2 \Gamma_{\Delta'}(\ell_2)$$

$$= \sum_{\ell_1 < z_2} \chi_{d'}(\ell_1) \Gamma_{\Delta'}(\ell_1) \mathscr{I}(z_2 - \ell_1, \Delta'; d'), \quad \text{say},$$

of $\mathscr{J}_4(z_2, \Delta'; d')$, the Dirichlet's series

$$F(s) = F_{\Delta', d'}(s) = \sum_{\ell=1}^{\infty} \frac{\chi_{d'}(\ell) \Gamma_{\Delta'}(\ell)}{\ell^s}$$

will also be seen to be the developing agent for the outer sum after it has been used to evaluate $\mathscr{I}(v, \Delta'; d')$. For $\sigma > 1$, definition (60) implies that[3])

$$(75) \quad F(s) = \prod_{p \nmid 2\Delta' d'} \left\{ 1 + \left(1 + \frac{1}{\theta_2(p)}\right) \frac{1}{p^s} \left(1 - \frac{1}{p^s}\right)^{-1} \right\} \prod_{p \mid 2\Delta', p \nmid d'} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$= L(s, \chi_{d'}) \prod_{p \nmid 2\Delta' d'} \left(1 + \frac{1}{\theta_2(p) p^s}\right) = L(s, \chi_{d'}) F_{2\Delta' d'}^{(1)}(s), \quad \text{say},$$

while, for $\sigma > 0$,

$$F_{2\Delta' d'}^{(1)}(s) = \prod_{p > 2} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \prod_{p > 2} \left(1 + \frac{1}{\theta_2(p) p^s}\right) \left(1 - \frac{1}{p^{s+1}}\right) \prod_{\substack{p \mid \Delta' d' \\ p > 2}} \left(1 + \frac{1}{\theta_2(p) p^s}\right)^{-1}$$

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[3]) The condition $p \nmid d'$ in the second product on the line below is unnecessary but helpful.

$$= \zeta(s+1)\left(1-\frac{1}{2^{s+1}}\right) M(s)\,\Theta(\Delta' d', s), \quad \text{say}. \tag{76}$$

Also, for $\sigma > \frac{1}{2}$ in the first place,

$$M(s) = \prod_{p>2}\left(1-\frac{1}{p^{2s+2}}+\frac{1}{p^{s+2}}\frac{2+1/p}{1-2/p-1/p^2}-\frac{2+1/p}{p^{2s+3}(1-2/p-1/p^2)}\right) \tag{77}$$

$$= \prod_{p>2}\left(1-\frac{1}{p^{2s+2}}\right)\left\{1+\frac{2+1/p}{1-2/p-1/p^2}\,\frac{1}{p^{s+2}}\left(1+\frac{1}{p^{s+1}}\right)^{-1}\right\}$$

$$= \frac{N(s)}{\zeta(2s+2)}, \quad \text{say}.$$

This with (75) and (76) supplies an analytic continuation for $F(s)$ that is valid in any extension of the half-plane $\sigma \geqq -\frac{1}{2}$ for which $\zeta(2s+2)$ is zero-free; here $N(s)$ is not only regular and absolutely bounded for $\sigma > -\frac{3}{4}$ but annihilates any poles of $\Theta(\Delta' d', s)$. Therefore the integrand in the formula

$$\frac{1}{2}\mathscr{I}(v, \Delta'; d') = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{\Delta', d'}(s-1)\,\frac{v^{s+2}}{s(s+1)(s+2)}\,ds \quad (0 < v \leqq z_2;\ c > 2)$$

has a factor

$$\prod_{p \mid d'}\left(1-\frac{1}{p^{s-1}}\right)\zeta(s-1)$$

with residue $\phi(d')/d'$ at $s = 2$; furthermore, its factor

$$\prod_{p \mid d'}\left(1-\frac{1}{p^{s-1}}\right)\zeta(s-1)\,\zeta(s)$$

is regular at $s = 1$ save when $d' = 1$, in which case it has residue $\zeta(0)$. Consequently, if we set

$$C_7 = \prod_{p>2}\left(1+\frac{1}{\theta_2(p)\,p}\right), \tag{78}$$

$$\psi_2(n) = \prod_{p \mid n;\, p>2}\left(1+\frac{1}{\theta_2(p)\,p}\right)^{-1} = \prod_{p \mid n;\, p>2}\frac{\theta_2(p)}{p-2}$$

and

$$\varrho_{d'} = \begin{cases} 1, & \text{if } d' = 1, \\ 0, & \text{if } d' > 1, \end{cases}$$

then we conclude for $0 < v \leqq z_2$ that

$$(79)\quad \frac{1}{2}\mathscr{I}(v,\Delta';d') = \frac{1}{24}C_7\frac{\phi(d')}{d'}\psi_2(\Delta'd')v^4 + \frac{1}{12}\varrho_{d'}\zeta(0)M(0)\Theta(\Delta'd',0)v^3 + O\left\{\Delta'^{\varepsilon}d'^{\frac{3}{4}}v^{\frac{5}{2}}e^{-A'\sqrt{\log(v+2)}}\right\}$$

because (75), (76), and (77) imply

$$F_{d',\Delta'}(s-1) = O\left(\left|\frac{\zeta(s-1)\zeta(s)}{\zeta(2s)}\right| d'^{\frac{3}{4}}\Delta'^{\varepsilon}\right) = O\left((|t|+1)^{\frac{3}{2}}d'^{\frac{3}{4}}\Delta'^{\varepsilon}\right)$$

in a region of type $\frac{1}{2} - A'/\log(|t|+2) < \sigma < \frac{3}{4}$.

If (79) be substituted in (74) in company with the bound

$$\sum_{\ell \leqq w}\Gamma_{\Delta'}(\ell) = O(w),$$

we get

$$(80)\quad \frac{1}{2}\mathscr{I}_4(z_2,\Delta';d') = \frac{1}{24}C_7\frac{\phi(d')}{d'}\psi_2(\Delta'd')\sum_{\ell_1<z_2}(z_2-\ell_1)^4\chi_{d'}(\ell_1)\Gamma_{\Delta'}(\ell_1) + \frac{1}{12}\varrho_{d'}\zeta(0)M(0)\Theta(2\Delta'd',0)\sum_{\ell_1<z_2}(z_2-\ell_1)^3\chi_{d'}(\ell_1)\Gamma_{\Delta'}(\ell_1) + O\left\{\Delta'^{\varepsilon}d'^{\frac{3}{4}}z_2^{\frac{7}{2}}e^{-A'\sqrt{\log(z_2+2)}}\right\},$$

in which the two sums are estimated by the two formulae

$$(81)\quad \frac{1}{24}\sum_{\ell<u}(u-\ell)^4\chi_{d'}(\ell)\Gamma_{\Delta'}(\ell) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F_{d',\Delta'}(s)\frac{u^{s+4}}{s(s+1)(s+2)(s+3)(s+4)}ds,$$

$$(82)\quad \frac{1}{6}\sum_{\ell<u}(u-\ell)^3\chi_{d_1'}(\ell)\Gamma_{\Delta'}(\ell) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F_{d',\Delta'}(s)\frac{u^{s+3}}{s(s+1)(s+2)(s+3)}ds$$

that are valid for $0<u=z_2$ and $c>1$, the main point of difference from what went before being the reaction between the denominator s and the behaviour of $F_{d',\Delta'}(s)$ at $s=0$. To apply the second formula to the second term in the right of (79) when $d'=1$, the main fact needed is that the integrand then has a pole with residue $\frac{1}{24}C_7\psi_2(\Delta')u^4$ at $s=1$ so that the sum in (82) is

$$(83)\quad \frac{1}{24}C_7\psi_2(\Delta')u^4 + O\left(u^{\frac{13}{4}}\right).$$

Similarly, for any value of d', the pole of the integrand in (81) has residue

$$(84)\quad \frac{1}{120}C_7\frac{\phi(d')}{d'}\psi_2(\Delta'd')u^5$$

at $s=1$, although the behaviour at $s=0$ is more complicated than before. First, if d' have more than one prime factor, then

$$L(s,\chi_{d'}) = \prod_{p\mid d'}\left(1-\frac{1}{p^s}\right)\zeta(s)$$

has a zero of multiplicity greater than one at $s=0$ and the integrand is therefore regular there. Next, if $d'=\varpi$,

$$\frac{L(s,\chi_{d'})}{s} = \zeta(0)\log\varpi$$

at $s=0$, the residue being

$$\frac{1}{48}\zeta(0)\,M(0)\,\Theta(\Delta' d',0)\,u^4\log\varpi \tag{85}$$

by (76). Also, if $d'=1$, the integrand is

$$\left(\frac{1}{s^2}+\frac{\gamma}{s}+\cdots\right)\left(1-\frac{1}{2^{s+1}}\right)\zeta(s)\,M(s)\,\Theta(\Delta' d',s)\frac{u^{s+4}}{(s+1)(s+2)(s+3)(s+4)}$$

near $s=0$ and the residue at $s=0$ is therefore

$$\begin{aligned}&\zeta(0)\,M(0)\,\Theta(\Delta' d',0)\frac{u^4}{48}\left(\frac{\zeta'(0)}{\zeta(0)}+\frac{M'(0)}{M(0)}+\frac{\Theta'(\Delta' d',0)}{\Theta(\Delta' d',0)}\right.\\ &\qquad\left.-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\log 2+\log u+\gamma\right)\\ &=\zeta(0)\,M(0)\,\Theta(\Delta' d',0)\frac{u^4}{48}\left(\frac{\zeta'(0)}{\zeta(0)}+\frac{M'(0)}{M(0)}+\frac{\Theta'(\Delta' d',0)}{\Theta(\Delta' d',0)}-\frac{25}{12}+\log 2+\log u+\gamma\right).\end{aligned} \tag{86}$$

Finally, by the reasoning used in (79), the residual integral in all cases is

$$O\left\{\Delta'^{\varepsilon} d'^{\frac{3}{4}} u^{\frac{7}{2}} e^{-A'\sqrt{\log(u+2)}}\right\}. \tag{87}$$

Let us collect together the results in (80), (83), (84), (85), (86), and (87) to form an asymptotic formula for $\frac{1}{2}\mathscr{I}_4(z_2,\Delta';d')$, noting that the result for $z_2<1$ is correctly derived but trivial. In all situations there is a residual term

$$O\left\{\Delta'^{\varepsilon} d'^{\frac{3}{4}} z_2^{\frac{7}{2}} e^{-A'\sqrt{\log(z_2+2)}}\right\} \tag{88}$$

and, by (84), a main term

$$\frac{1}{120}C_7^2\frac{\phi^2(d')}{d'^2}\psi_2^2(\Delta d)\,z_2^{\frac{5}{2}}, \tag{89}$$

about which in fact all we shall need to know is that it is of the form $A(d',\Delta')z_2^5$. Also when $d'=\varpi$ there is an additional term

$$\frac{1}{48}C_7\zeta(0)M(0)\frac{\phi(\varpi)}{\varpi}\psi_2(\Delta d)\Theta(2\Delta d,0)z_2^4\log\varpi \tag{90}$$

while, if $d'=1$, there is instead the term

$$\frac{1}{48}C_7\zeta(0)M(0)\psi_2(\Delta d)\Theta(\Delta d,0)\left(\frac{\zeta'(0)}{\zeta(0)}+\frac{M'(0)}{M(0)}+\frac{\Theta'(\Delta d,0)}{\Theta(\Delta d,0)}\right.$$
$$\left.+\log z_2+\gamma-\frac{13}{12}+\log 2\right)z_2^4 \tag{91}$$

to be added.

The assessment of the constituent $\mathscr{J}_5(z_2,\Delta';\chi_{d'}^*)$ in the formula (73) for $\mathscr{J}_6(z_2,\Delta';d')$ is based on the generating functions

$$F_{\Delta'}(s,\chi_{d'}^*)=\sum_{\ell=1}^{\infty}\frac{\chi_{d'}^*(\ell)\Gamma_{\Delta'}(\ell)}{\ell^s}$$

and $F_{\Delta'}(s,\bar{\chi}_{d'}^*)$, the former by analogy with the formulae for $F_{\Delta',d'}(s)$ being equal through analytic continuation to the last element in the chain of equations

$$L(s,\chi_{d'}^*)\prod_{p\nmid 2\Delta'}\left(1+\frac{\chi_{d'}^*(p)}{\theta_2(p)p^s}\right) \tag{92}$$
$$=L(s,\chi_{d'}^*)L(s+1,\chi_{d'}^*)\prod_{p\mid 2\Delta}\left(1-\frac{\chi_{d'}^*(p)}{p^{s+1}}\right)\prod_{p\nmid 2\Delta}\left(1+\frac{\chi_{d'}^*(p)}{\theta_2(p)p^s}\right)\left(1-\frac{\chi_{d'}^*(p)}{p^{s+1}}\right)$$
$$=L(s,\chi_{d'}^*)L(s+1,\chi_{d'}^*)X(s)$$

where

$$X(s)=X(s,\Delta';\chi_{d'}^*)=O(\Delta'^{\varepsilon})\quad\left(\sigma\geqq-\frac{1}{4}\right). \tag{93}$$

To direct this to the sums arising from the counterpart

$$\mathscr{J}_5(z_2,\Delta';\chi_{d'}^*)=\sum_{\ell_1<z_2}\chi_{d'}^*(\ell_1)\Gamma_{\Delta'}(\ell_1)\sum_{\ell_2<z_2-\ell_1}(z_2-\ell_1-\ell_1)^2\bar{\chi}_{d'}^*(\ell_2)\ell_2\Gamma_{\Delta'}(\ell_2) \tag{94}$$
$$=\sum_{\ell_1<z_2}\chi_{d'}^*(\ell_1)\Gamma_{\Delta'}(\ell_1)\mathscr{I}(z_2-\ell_1,\Delta;\bar{\chi}_{d'}^*),\quad\text{say},$$

of (74), the only substantial property of the Dirichlet's L-functions needed is

Lemma 4. *For a Dirichlet's L-function formed with a non-principal character* χ, mod q, *we have*

$$L(s,\chi)=\begin{cases} 0\left(\{(|t|+1)q\}^{\frac{1}{2}(1-\sigma)+\varepsilon}\right), & \text{if } 0\leqq\sigma\leqq 1,\\ 0\left(\{(|t|+1)q\}^{\frac{1}{2}-\sigma+\varepsilon}\right), & \text{if } \sigma<0.\end{cases}$$

First

$$\frac{1}{2}\mathscr{I}(v,\Delta';\bar{\chi}^*_{d'})=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F_{\Delta'}(s,\bar{\chi}^*_{d'})\frac{v^{s+3}}{(s+1)(s+2)(s+3)}ds\quad(c>1)$$

$$=\frac{1}{2\pi i}\int_{-\eta-i\infty}^{-\eta+i\infty}F_{\Delta'}(s,\bar{\chi}^*_{d'})\frac{v^{s+3}}{(s+1)(s+2)(s+3)}ds,$$

since the initially valid choice for c can be reduced to $-\eta=-\frac{1}{8}$ by Lemma 4 and the regularity of the integrand for $\sigma\geqq-\eta$. Secondly, by (94),

$$\mathscr{I}_5(z_2,\Delta';\chi^*_{d'})=\frac{1}{2\pi i}\int_{-\eta-i\infty}^{-\eta+i\infty}\frac{F_{\Delta'}(s,\bar{\chi}^*_{d'})}{(s+1)(s+2)(s+3)}\sum_{\ell<z_2}(z_2-\ell)^{s+3}\chi^*_{d'}(\ell)\Gamma_{\Delta'}(\ell)\,ds;$$

here a well-known formula in the calculus of residues shews that the sum in the integrand equals

$$=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}F_{\Delta'}(s',\chi^*_{d'})\frac{\Gamma(s+4)\Gamma(s')}{\Gamma(s+s'+4)}z_2^{s+s'+3}\,ds'\quad(c'>1)$$

$$=z_2^{s+3}F_{\Delta'}(0,\chi^*_{d'})+\frac{1}{2\pi i}\int_{-\eta-i\infty}^{-\eta+i\infty}F_{\Delta'}(s',\chi^*_{d'})\frac{\Gamma(s+4)\Gamma(s')}{\Gamma(s+s'+4)}z_2^{s+s'+3}\,ds',$$

where at this point in accordance with the conventions laid down in §2 the symbol $\Gamma(s)$ denotes the Gamma-function and not the arithmetical function defined in (60). Therefore

$$(95)\quad \mathscr{I}_5(z_2,\Delta';\chi^*_{d'})=\frac{1}{2\pi i}F_{\Delta'}(0,\chi^*_{d'})\int_{-\eta-i\infty}^{-\eta+i\infty}F_{\Delta'}(s,\bar{\chi}^*_{d'})\frac{z_2^{s+3}}{(s+1)(s+2)(s+3)}ds$$

$$-\frac{1}{4\pi^2}\int_{-\eta-i\infty}^{-\eta+i\infty}\int_{-\eta-i\infty}^{-\eta+i\infty}F_{\Delta'}(s,\bar{\chi}^*_{d'})F_{\Delta'}(s',\chi^*_{d'})\frac{\Gamma(s+1)\Gamma(s')}{\Gamma(s+s'+4)}z_2^{s+s'+3}\,ds\,ds',$$

to estimate which we need the order relation

$$\Gamma(s'')\asymp(|t''|+1)^{\sigma''-\frac{1}{2}}e^{-\frac{1}{2}\pi|t''|}\quad(|\sigma''|>\eta_1)$$

that comes from Stirling's theorem. If $\sigma=\sigma'=-\eta$ throughout, this gives

$$(96)\qquad \frac{\Gamma(s+1)\Gamma(s')}{\Gamma(s+s'+4)}=O\left(\frac{(|t|+1)^{\frac{1}{2}-\eta}(|t'|+1)^{-\frac{1}{2}-\eta}}{(|t|+|t'|+1)^{\frac{7}{2}-2\eta}}\right)$$

$$=O\left(\frac{1}{(|t|+1)^{\frac{7}{4}}(|t'|+1)^{\frac{7}{4}}}\right)$$

when t and t' are of the same sign. On the other hand, in the contrary instance, we may consider the typical cases where $|t'| > 2t$ and where $|t| \leqq |t'| \leqq 2|t|$, in the former of which (96) still holds because $|t' - t| > |t|, \frac{1}{2}|t'|$; but in the latter case, being

$$O\left\{(|t|+1)^{\frac{1}{2}-\eta}(|t'|+1)^{-\frac{1}{2}-\eta}e^{-\frac{1}{2}\pi|t|}\right\},$$

the left-side of (96) is still obviously subject to the estimate in its right-side. Consequently, by (95), (96), and Lemma 4,

$$(97)\qquad \mathscr{J}_5(z_2, \Delta'; \chi_{d'}^*) = O\left(\Delta'^{\varepsilon} d'^{1+\frac{3}{2}\eta+\varepsilon} z_2^{3-\eta}\right)$$

$$+ O\left\{\Delta'^{\varepsilon} d'^{1+3\eta+\varepsilon} z_2^{3-2\eta} \int_0^{\infty}\int_0^{\infty} \frac{dt\,dt'}{(t+1)^{\frac{5}{4}-\frac{3\eta}{2}}(t'+1)^{\frac{5}{4}-\frac{3\eta}{2}}}\right\}$$

$$= O\left(\Delta'^{\varepsilon} d'^{1+\frac{3\eta}{2}+\varepsilon} z_2^{3-\eta}\right) + O(\Delta'^{\varepsilon} d'^{1+3\eta+\varepsilon} z_2^{3-2\eta});$$

a better estimate could be obtained by using mean-value theorems for L-functions but would not confer any greater benefits for our present investigation.

All is in place for the production of the earlier formula for $\mathscr{J}_2^*(z_1, \Delta; d)$. First, by (73) and (97), the contribution of $\mathscr{J}_5(z_1/e, \Delta'; \chi_{d'}^*)$ to $\mathscr{J}_2^*(z_1, \Delta; d)$ via (72) is

$$(98)\quad O\left(\Delta^{\varepsilon} d^{1+\frac{3}{2}\eta+\varepsilon} z_1^{3-\eta} \sum_{e|d} \frac{\Gamma_{\Delta}^2(e)}{e^{1+\frac{1}{2}\eta}}\right) + O\left(\Delta^{\varepsilon} d^{1+3\eta+\varepsilon} z_1^{3-2\eta} \sum_{e|d} \frac{\Gamma_{\Delta}^2(e)}{e^{1+\eta}}\right)$$

$$= O\left(\Delta^{\varepsilon} d^{1+\frac{3}{2}\eta+\varepsilon} z_1^{3-\eta}\right) + O\left(\Delta^{\varepsilon} d^{1+3\eta+\varepsilon} z_1^{3-2\eta}\right).$$

Also, by some minor calculations, we see that the combined effect of (88), (89), (90), and (91) on $\mathscr{J}_2^*(z_1, \Delta; d)$ in (72) is

$$(99)\qquad H_1(\Delta, d) z_1^5 + \frac{C_7 \zeta(0) M(0) \psi_2(\Delta d) \Theta(\Delta d, 0) \Gamma^2(d) z_1^4}{12 d} \sum_{\varpi | d} \frac{\log \varpi}{\Gamma^2(\varpi)}$$

$$+ \frac{C_7 \zeta(0) M(0) \psi_2(\Delta d) \Theta(\Delta d, 0) \Gamma^2(d) z_1^4}{12 d} \left(\frac{\zeta'(0)}{\zeta(0)} + \frac{M'(0)}{M(0)} + \frac{\Theta'(\Delta d, 0)}{\Theta(\Delta d, 0)}\right.$$

$$\left.+ \log\frac{z_1}{d} + \gamma - \frac{13}{12} + \log 2\right) + O\left(\frac{z_1^{\frac{7}{2}} d^{\frac{3}{4}} \Delta^{\varepsilon}}{\phi(d)} \sum_{e|d} \frac{\Gamma_{\Delta}^2(e)}{e^{\frac{1}{4}}} e^{-A'\sqrt{\log\{(z_1/e)+2\}}}\right)$$

$$= H_1(\Delta, d) z_1^5 + H_2(\Delta, d) z_1^4 \log z_1 + H_3(\Delta, d) z_1^4 + O\left(\Delta^{\varepsilon} d^{-\frac{1}{4}} z_1^{\frac{7}{2}} e^{-A'\sqrt{\log(z_1+2)}}\right), \quad \text{say},$$

where $H_1(\Delta, d)$ has its provenance in the function $A(d', \Delta')$ appearing implicitly in (89). Thus $\mathscr{J}_2^*(z_1, \Delta; d)$ is the sum of the expressions in the final terms of (98) and (99).

Lastly, having found our first formula for $\mathscr{J}_2^*(z_1, \Delta; d)$, we can comment on our choosing to make $\mathscr{J}_2(z, \Delta)$ the first object of the treatment. A barrier to a simple analysis of the sum $\mathscr{J}_1^*(z_1, \Delta; d)$ in (67) is the presence of the denominator ℓ that precludes effective iterated summations over ℓ_1 and ℓ_2 and that necessitates instead some unwelcome device such as the wholesale use of double contour integrals. Although the most obvious method of overcoming this difficulty would be simply to remove the denominator ℓ and to use (69) in place of the consequential Lemma 3, the calculations parallel to those we employed would become onerous in connection with multiple poles occurring in the integrands; however, by changing the rôle of ℓ from denominator to numerator, we see that the first and second integrations are considerably simplified.

8. Estimation of $\mathscr{J}_2^*(z_1, \Delta; d)$ – second method

In the second procedure for calculating $\mathscr{J}_2^*(z_1, \Delta; d)$ we deal directly with the inner sum

$$\Omega_\Delta(\ell) = \sum_{\ell_1 + \ell_2 = \ell} \Gamma_\Delta(\ell_1)\,\Gamma_\Delta(\ell_2)$$

that is contained in its implicit definition by (68) as a double sum.

Since

$$\Gamma(L) = \sum_{\substack{mr = L \\ (m, 2\Delta) = 1}} \frac{\mu^2(m)}{\theta_2(m)}$$

by (60), we have

$$\Omega_\Delta(\ell) = \sum_{\substack{m_1, m_2 < \ell \\ (m_1, m_2, 2\Delta) = 1}} \frac{\mu^2(m_1)\,\mu^2(m_2)\,v(m_1, m_2, \ell)}{\theta_2(m_1)\,\theta_2(m_2)},$$

where $v(m_1, m_2, \ell)$ is the number of positive solutions in r_1, r_2 of the indeterminate equation $m_1 r_1 + m_2 r_2 = \ell$ and where therefore

$$v(m_1, m_2, \ell) = \begin{cases} \ell/[m_1, m_2] + O(1), & \text{if } (m_1, m_2) | \ell, \\ 0, & \text{otherwise}. \end{cases}$$

Consequently, if we write

$$\Omega_\Delta^*(\ell) = \sum_{\substack{(m_1 m_2, 2\Delta) = 1 \\ (m_1, m_2) | \ell}} \frac{\mu^2(m_1)\,\mu^2(m_2)\,(m_1, m_2)}{\theta_2(m_1)\,\theta_2(m_2)\,m_1 m_2}$$

for convenience, then[4])

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[4]) It is helpful to keep the condition $2 \nmid m_1 m_2$ in what follows because $\theta_2(2)$ is negative.

$$\Omega_\Delta(\ell) - \ell\Omega_\Delta^*(\ell) = O\Bigg(\sum_{\substack{m_1, m_2 < \ell \\ 2 \nmid m_1 m_1}} \frac{\mu^2(m_1)\mu^2(m_2)}{\theta_2(m_1)\theta_2(m_2)}\Bigg) + O\Bigg(\ell \sum_{\substack{m_1 \geqq \ell \\ 2 \nmid m_1 m_2}} \frac{\mu^2(m_1)\mu^2(m_2)(m_1, m_2)}{\theta_2(m_1)\theta_2(m_2) m_1 m_2}\Bigg)$$

$$= O(\log^2 \ell) + O\Bigg(\ell \sum_{\substack{m_1 \geqq \ell \\ 2 \nmid m_1}} \frac{\mu^2(m_1)}{\theta_2(m_1) m_1} \sum_{2 \nmid m_2} \frac{\mu^2(m_2)(m_1, m_2)}{\theta_2(m_2) m_2}\Bigg)$$

$$= O(\log^2 \ell) + O\Bigg(\ell \sum_{\substack{m_1 \geqq \ell \\ 2 \nmid m}} \frac{\mu^2(m_1)\sigma_{-1}(m_1)}{\theta_2(m_1) m_1}\Bigg)$$

$$= O(\log^2 \ell) + O(1) = O(\log^2 2\ell) \tag{100}$$

by elementary calculations implicating divisor-type functions. Also, exploiting the generalization of Euler's theorem to series containing multiplicative functions $f(m_1, m_2)$ such that $f(m_1' m_1'', m_2' m_2'') = f(m_1', m_2') f(m_1'', m_2'')$ when $(m_1' m_2', m_1'' m_2'') = 1$, we deduce that

$$\Omega_\Delta(\ell) = \prod_{\substack{p \nmid 2\Delta \\ p \nmid \ell}} \left(1 + \frac{2}{\theta_2(p) p}\right) \prod_{\substack{p \nmid 2\Delta \\ p \mid \ell}} \left(1 + \frac{2}{\theta_2(p) p} + \frac{1}{\theta_2^2(p) p}\right),$$

from which and the equality

$$1 + \frac{2}{\theta_2(p) p} = \frac{(p-1)^2}{\theta_2(p) p} \tag{101}$$

it follows that

$$\Omega_\Delta^*(\ell) = \prod_{p \nmid 2\Delta} \left(1 + \frac{2}{\theta_2(p) p}\right) \prod_{\substack{p \nmid 2\Delta \\ p \mid \ell}} \left(1 + \frac{1}{\theta_2(p)(p-1)^2}\right) \tag{102}$$

$$= \frac{C_8 U(\Delta) K(\ell)}{K\{(\ell.\Delta)\}}$$

where

$$\left\{\begin{aligned} C_8 &= \prod_{p > 2} \left(1 + \frac{2}{\theta_2(p) p}\right), \\ U(\Delta) &= \prod_{\substack{p \mid \Delta \\ p > 2}} \left(1 + \frac{2}{\theta_2(p)}\right)^{-1} = \prod_{\substack{p \mid \Delta \\ p > 2}} \frac{p\theta_2(p)}{(p-1)^2}, \\ \text{and} \quad K(\ell) &= \prod_{\substack{p \mid \ell \\ p > 2}} \left(1 + \frac{1}{\theta_2(p)(p-1)^2}\right). \end{aligned}\right. \tag{103}$$

In all, we therefore find that

$$\Omega_\Delta(\ell) = \frac{C_8 U(\Delta) K(\ell) \ell}{K\{(\ell, \Delta)\}} + O(\log^2 2\ell) \tag{104}$$

from (100) and (102).

By way of (68) and (104) we return to $\mathscr{J}_2^*(z_1, \Delta; d)$ and infer that

$$(105)\quad \mathscr{J}_2^*(z_1, \Delta; d) = C_8\, U(\Delta) \sum_{\substack{\ell < z_1 \\ \ell \equiv 0,\, \mathrm{mod}\, d}} (z_1 - \ell)^2 \frac{K(\ell)\,\ell^2}{K\{(\ell, \Delta)\}} + O\left(\frac{z_1^4 \log^2 2z_1}{d}\right)$$

$$= C_8\, U(\Delta)\, \mathscr{J}_7(z_1, \Delta; d) + O\left(\frac{z_1^4 \log^2 2z_1}{d}\right), \quad \text{say},$$

the generating function for the sum $\mathscr{J}_7(z_1, \Delta; d)$ being the Dirichlet's series

$$\sum_{\ell \equiv 0,\, \mathrm{mod}\, d} \frac{K(\ell)}{K\{(\ell, \Delta)\}\,\ell^s} = \frac{1}{d^s}\left(1 - \frac{1}{2^s}\right) \prod_{p \mid d} K(p)\left(1 - \frac{1}{p^s}\right)^{-1} \prod_{\substack{p \mid \Delta \\ p > 2}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$\times \prod_{\substack{p \nmid \Delta \\ p > 2}} \left\{1 + \frac{K(p)}{p^s}\left(1 - \frac{1}{p^s}\right)^{-1}\right\}$$

$$= \frac{K(d)\,\zeta(s)}{d^s} \prod_{\substack{p \nmid \Delta \\ p > 2}} \left(1 + \frac{1}{p^s}\{K(p) - 1\}\right)$$

$$(106)\quad = \frac{1}{d^s} K(d)\,\zeta(s)\,G_\Delta(s), \quad \text{say},$$

where (103) shews that $G_\Delta(s)$ represents an absolutely bounded regular function for $\sigma \geqq -\frac{3}{2}$. Hence, in the familiar way,

$$\mathscr{J}_7(z_1, \Delta; d) = 2K(d) \int_{c - i\infty}^{c + i\infty} \zeta(s)\, G_\Delta(s) \frac{z_1^{s+4}}{d^s (s+2)(s+3)(s+4)}\, ds$$

for $c > 1$ in the first place, wherefore, by moving the line of integration to $\sigma = -\frac{5}{4}$, we obtain

$$(107)\quad \mathscr{J}_7(z_1, \Delta; d) = \frac{1}{30\,d} K(d)\, G_\Delta(1)\, z_1^5 + O\left(d^{\frac{5}{4}} z_1^{\frac{11}{4}}\right).$$

Thus, combining this with (105), we complete the second assessment by concluding that

$$(108)\quad \mathscr{J}_2^*(z_1, \Delta; d) = H_1^{(1)}(\Delta, d)\, z_1^5 + O\left(\frac{z_1^4 \log^2 2z_1}{d}\right) + O\left(d^{\frac{5}{4}} z_1^{\frac{11}{4}}\right)$$

for a suitable function $H_1^{(1)}(\Delta, d)$.

9. Treatments of $\mathscr{J}_2(z_1, \Delta)$ and $\mathscr{J}_1(z, \Delta)$

To synthesize an effective formula for $\mathscr{J}_2^*(z_1, \Delta; d)$ from those thus far obtained, we first compare (98) and (99) with (108) for any given values of Δ and d and deduce that

$$H_1(\Delta, d) = H_1^{(1)}(\Delta, d)$$

by letting $z_1 \to \infty$. Thus, by part of the calculation that led from (107) to (108),

$$(109) \qquad H_1(\Delta, d) z_1^5 = C_8 U(\Delta) \mathscr{J}_7(z_1, \Delta; d) + O\left(d^{\frac{5}{4}} z_1^{\frac{11}{4}}\right)$$

so that (98) and (99) yield

$$(110) \quad \mathscr{J}_2^*(z_1, \Delta; d) = C_8 U(\Delta) \mathscr{J}_7(z_1, \Delta; d) + H_2(\Delta, d) z_1^4 \log z_1 + H_3(\Delta, d) z_1^4$$
$$+ O\left(\Delta^\varepsilon d^{\frac{19}{16}+\varepsilon} z_1^{\frac{23}{8}}\right) + O\left(\Delta^\varepsilon d^{\frac{11}{8}+\varepsilon} z_1^{\frac{11}{4}}\right) + O\left(\Delta^\varepsilon d^{-\frac{1}{4}} z_1^{\frac{7}{2}} e^{-A\sqrt{\log(z_1+2)}}\right)$$

when $\eta = \frac{1}{8}$ as before. Both this formula and its variant (108) being valid for $1 \leqq z_1 \leqq z$ not only when $d \leqq z_1$ but even when $d \leqq z$, an examination of $H_2(\Delta, d)$ and $H_3(\Delta, d)$ with the aid of (99), (76), and (78) shews that the second and third explicit terms on the right of (110) may be conveniently added to the right of (108) provided that the $\log^2 2z_1$ appearing in the latter be replaced by z^ε. But it is easily confirmed that

$$\min\left(d^{\frac{19}{16}} z_1^{\frac{23}{8}} + d^{\frac{11}{8}} z^{\frac{11}{4}}, z_1^4/d\right) = \min\left(d^{\frac{19}{16}} z_1^{\frac{23}{8}}, z_1^4/d\right),$$

where it is critical to the method that z_1^4/d only take the smaller value in a range of d commencing at a value $z_1^{\frac{18}{35}}$ that is significantly larger that $z_1^{\frac{1}{2}}$. In summary, therefore, we have the formula

$$(111) \quad \mathscr{J}_2^*(z_1, \Delta; d) = C_8 U(\Delta) \mathscr{J}_7(z_1, \Delta; d) + H_2(\Delta, d) z_1^4 \log z_1 + H_3(\Delta, d) z_1^4$$
$$+ O\left\{\Delta^\varepsilon z^\varepsilon \min\left(d^{\frac{19}{16}} z_1^{\frac{23}{8}}, z_1^4/d\right)\right\} + O\left(\Delta^\varepsilon d^{-\frac{1}{4}} z_1^{\frac{7}{2}} e^{-A\sqrt{\log(z_1+2)}}\right).$$

Ere we continue our climb, it is appropriate to pause awhile to look back on our recent route. The inadequacy of the first method in §7 for larger values of d is actually related to the estimation of $\mathscr{J}_6(z, \Delta'; d')$ in (73), which notwithstanding initial impressions actually contributes in all a term of significant size to the expression being evaluated. To best appreciate this effect we remark that, had we been working directly with $\mathscr{J}_1(s, \Delta)$ instead of with $\mathscr{J}_2(s, \Delta)$, the analysis of the counterpart of $\mathscr{J}_5(z_2, \Delta'; \chi_{d'}^*)$ through a double contour integral would have, *inter alia*, thrown up an explicit term containing $|L(0, \chi_{d'}^*)|^2$, the summation of which over $\chi_{d'}^*$ would not involve cancellation because $L(0, \chi_{d'}^*) = 0$ when $\chi_{d'}^*(-1) = 1$. But, as our procedure demonstrates, this apparently awkward donation to the work can be accounted for by substituting $C_8 U(\Delta) \mathscr{J}_7(z_1, \Delta; d)$ for $H_1(\Delta, d) z_1^5$, which action additionally means that difficulties associated with the consequential summation over d are circumvented by the use of (112) and (113) below.

In preparation for the evaluation of $\mathscr{J}_1(z, \Delta)$, we make a nominal change in the range of summation over d (still square-free as in (70)) in (68) to give the equivalent representation of $\mathscr{J}_2(z_1, \Delta)$ as

$$\sum_{\substack{d \leqq z \\ (d, 2\Delta) = 1}} \frac{\mathscr{J}_2^*(z_1, \Delta; d)}{\theta_2(d)} \quad (z_1 \geqq 1; z \geqq z_1)$$

and then temporarily agree for clarity to let the symbol $O^z(f)$ denote a quantity that is $O(f)$ and that is independent of z. We then see from (111), (105), and (60) that

$$\mathscr{J}_2(z_1, \Delta) = C_8 U(\Delta) \sum_{\ell < z_1} (z_1 - \ell)^2 \frac{\Gamma_\Delta(\ell) K(\ell) \ell^2}{K\{(\ell, \Delta)\}} + z_1^4 \log z_1 \sum_{d \leqq z} \frac{H_2(\Delta, d)}{\theta_2(d)} + z_1^4 \sum_{d \leqq z} \frac{H_3(\Delta, d)}{\theta_2(d)}$$

$$+ \sum_{d \leqq z_1^{18/35}} O^z\left(\frac{\Delta^\varepsilon z_1^{\frac{23}{8}+\varepsilon} d^{\frac{19}{16}}}{\theta_2(d)}\right) + \sum_{d > z_1^{18/35}} O^z\left(\frac{\Delta^\varepsilon z_1^{4+\varepsilon}}{d\theta_2(d)}\right)$$

$$+ \sum_d O^z\left(\frac{\Delta^\varepsilon z_1^{\frac{7}{2}} e^{-\sqrt{\log(z_1+2)}}}{d^{\frac{1}{4}}\theta_2(d)}\right) + O\left(\Delta^\varepsilon z_1^{4+\varepsilon} \sum_{d > z} \frac{1}{d\theta_2(d)}\right)$$

$$+ O\left(\Delta^\varepsilon z_1^{\frac{7}{2}} e^{-\sqrt{\log(z_1+2)}} \sum_{d > z} \frac{1}{d^{\frac{1}{4}}\theta_2(d)}\right),$$

within which the remainder terms amount to

$$O^z\left(\Delta^\varepsilon z_1^{\frac{122}{35}}\right) + O^z\left(\Delta^\varepsilon z_1^{\frac{7}{2}} e^{-\sqrt{\log(z_1+2)}}\right) + O\left(\frac{\Delta^\varepsilon z_1^{4+\varepsilon}}{z}\right) + O\left(\frac{\Delta^\varepsilon z_1^{\frac{7}{2}} e^{-\sqrt{\log(z_1+2)}}}{z^{\frac{1}{4}}}\right)$$

$$= O^z\left(\Delta^\varepsilon z_1^{\frac{7}{2}} e^{-\sqrt{\log(z_1+2)}}\right) + O\left(\frac{\Delta^\varepsilon z_1^{\frac{7}{2}}}{z^{\frac{1}{4}}}\right).$$

Hence

$$\mathscr{J}_2(z_1, \Delta) = C_8 U(\Delta) \sum_{\ell < z_1} (z_1 - \ell)^2 \frac{\Gamma_\Delta(\ell) K(\ell) \ell^2}{K\{(\ell, \Delta)\}} + z_1^4 H_2^*(z, \Delta) \log z_1$$
$$+ z_1^4 H_3^*(z, \Delta) + O\left(\frac{\Delta^\varepsilon z_1^{\frac{7}{2}}}{z^{\frac{1}{4}}}\right) + O^z\left(\Delta^\varepsilon z_1^{\frac{7}{2}} e^{-\sqrt{\log(z_1+2)}}\right), \quad \text{say}. \tag{112}$$

Let us now insert this expression for $\mathscr{J}_2(z_1, \Delta)$ into the formula for $\mathscr{J}_1(z, \Delta)$ given by Lemma 3. The impact of the first term on the right side of (112) is clearly

$$C_8 U(\Delta) \sum_{\ell < z} (z - \ell)^2 \frac{\Gamma_\Delta(\ell) K(\ell)}{K\{(\ell, \Delta)\}} = V_1(z, \Delta), \quad \text{say}, \tag{113}$$

while that of the other terms is

$$z^2 \log z H_2^*(z, \Delta) + z^2 H_3^*(z, \Delta) - 6 z H_2^*(z, \Delta) \int_1^z \log u \, du - 6 z H_3^*(z, \Delta) \int_1^z du$$

$$+ 12 z^2 H_2^*(z, \Delta) \int_1^z \frac{\log u \, du}{u} + 12 z^2 H_3^*(z, \Delta) \int_1^z \frac{du}{u}$$

$$+ O\left(\Delta^\varepsilon z^{\frac{3}{2}} e^{-A'\sqrt{\log(z+2)}}\right) + O\left(\Delta^\varepsilon z \int_1^z \frac{e^{-A'\sqrt{\log(u+2)}}}{u^{\frac{1}{2}}} du\right)$$

$$+ 12 z^2 \int_1^\infty O^z\left(\frac{\Delta^\varepsilon e^{-A'\sqrt{\log(u+2)}}}{u^{\frac{3}{2}}}\right) du + O\left(z^2 \Delta^\varepsilon \int_z^\infty \frac{e^{-A'\sqrt{\log(u+2)}}}{u^{\frac{3}{2}}} du\right) + O\left(z^{\frac{7}{4}} \Delta^\varepsilon \int_1^\infty \frac{du}{u^{\frac{3}{2}}}\right)$$

$$= 6 z^2 \log^2 z H_2^*(z, \Delta) + z^2 \log z \{-5 H_2^*(z, \Delta) + 12 H_3^*(z, \Delta)\}$$

$$+ z^2 \{6 H_2^*(z, \Delta) - 5 H_3^*(z, \Delta) + H_4^*(\Delta)\} + O\left(\Delta^\varepsilon z^{\frac{3}{2}} e^{-A'\sqrt{\log(z+2)}}\right), \quad \text{say},$$

wherein

$$H_4^*(\Delta) = O(\Delta^\varepsilon). \tag{114}$$

The only constituents in these formulae that are not in readiness for their use in formula (64) for $\mathscr{J}(y)$ are the first two terms in the antepenultimate line, which when transformed for $z = y/\Delta$ and combined with the following terms produce

$$(115) \quad 6 y^2 \log^2 y \frac{H_2^*(y/\Delta, \Delta)}{\Delta^2} + \frac{y^2 \log y}{\Delta^2} \{-5 H_2^*(y/\Delta, \Delta) + 12 H_3^*(y/\Delta, \Delta)$$

$$- 12 H_2^*(y/\Delta, \Delta) \log \Delta\} + \frac{y^2}{\Delta^2} \{6 H_2^*(y/\Delta, \Delta) \log^2 \Delta + 5 H_2^*(y/\Delta, \Delta) \log \Delta$$

$$- 12 H_3^*(y/\Delta, \Delta) \log \Delta + 6 H_2^*(y/\Delta, \Delta) - 5 H_3^*(y/\Delta, \Delta) + H_4(\Delta)\}$$

$$+ O\left(\frac{\Delta^\varepsilon y^{\frac{3}{2}}}{\Delta^{\frac{3}{2}}} e^{-A'\sqrt{\log\{(y/\Delta)+2\}}}\right)$$

$$= y^2 \log^2 y V_2(y/\Delta, \Delta) + y^2 \log y V_3(y/\Delta, \Delta) + y^2 V_4(y/\Delta, \Delta)$$

$$+ O\left(\frac{\Delta^\varepsilon y^{\frac{3}{2}}}{\Delta^{\frac{3}{2}}} e^{-A'\sqrt{\log(y/\Delta+2)}}\right), \quad \text{say}.$$

Then, in anticipation of the estimation of $\mathscr{J}(y)$ that is to follow, we write

$$W_i(y) = \sum_{\Delta \leq y} \frac{\mu(\Delta) \Delta E^\dagger(\Delta) \tau(\Delta)}{\theta_2(\Delta)} V_i(y/\Delta, \Delta) \tag{116}$$

in order to get the decomposition

$$(117)\qquad \mathscr{J}(y) = W_1(y) + y^2\log^2 y\, W_2(y) + y^2 \log y\, W_3(y) + y^2 W_4(y) + O\left(y^{\frac{3}{2}} \sum_{\Delta < y} \frac{1}{\Delta^{\frac{5}{4}}} e^{-A'\sqrt{\log(y/\Delta + 2)}}\right)$$

$$= W_1(y) + y^2\log^2 y\, W_2(y) + y^2\log y\, W_3(y) + y^2 W_4(y) + O\left(y^{\frac{3}{2}} e^{-A'\sqrt{\log(y+2)}}\right)$$

that flows from (65), (59), (66), and (115).

Finally, we should remark at once that the complication in the form of $V_4(y/\Delta, \Delta)$ is not reflected in the estimation of the corresponding sum $W_4(y)$, which can be quickly dismissed and has little influence on the final outcome.

10. Estimation of $\mathscr{J}(y)$

The summit is almost in sight because all information needed for our theorem will be available once the sums $W_i(y)$ have been assessed.

Pointing (116) and (113) at the primary sum $W_1(y)$, we have

$$(118)\qquad W_1(y) = C_8 \sum_{\Delta\ell < y} (y - \Delta\ell)^2 \frac{\mu(\Delta)\, U(\Delta)\, E^{\dagger}(\Delta)\, \tau(\Delta)\, \Gamma_{\Delta}(\ell)\, K(\ell)}{\Delta\, \theta_2(\Delta)\, K\{(\ell, \Delta)\}}$$

$$= C_8 \sum_{n<y} (y-n)^2 a_n, \quad \text{say},$$

in which, being seen to be multiplicative, the coefficient a_n is equal for an odd prime power p^{α} to

$$\Gamma(p)\, K(p) - \frac{U(p)\, \tau(p)}{p\theta_2(p)}$$

$$= \left(1 + \frac{1}{\theta_2(p)}\right)\left(1 + \frac{1}{\theta_2(p)(p-1)^2}\right) - \frac{1}{(p-1)^2}\left(1 + \frac{3}{\theta_2(p)} + \frac{1}{\theta_2^2(p)}\right)$$

$$= 1 - \frac{1}{(p-1)^2} + \frac{1}{\theta_2(p)} - \frac{2}{\theta_2(p)(p-1)^2} = 1 - \frac{1}{(p-1)^2} + \frac{p}{(p-1)^2} = \frac{p}{p-1}$$

in virtue of (60). Since also $a_n = 2$ when n is a power 2^{α}, the generating function of the sum in (118) is

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \left\{1 + \frac{2}{2^s}\left(1 - \frac{1}{2^s}\right)^{-1}\right\} \prod_{p>2} \left\{1 + \frac{p}{(p-1)p^s}\left(1 - \frac{1}{p^s}\right)^{-1}\right\}$$

$$= \zeta(s)\left(1 + \frac{1}{2^s}\right)\left(1 + \frac{1}{2^s}\right)^{-1} \prod_p \left(1 + \frac{1}{(p-1)p^s}\right)$$

$$= \zeta(s)\, \zeta(s+1)\, h(s)$$

in the notation of the lemmata numbered 1 both here and in I. Hence, by following earlier procedures as exemplified in the proof of the above-mentioned lemmata and by recalling that $h(0)=1$, we find that

$$W_1(y) = \frac{1}{3} C_8 \frac{\zeta(2)\zeta(3)}{\zeta(6)} y^3 + C_8 \zeta(0) y^2 \log y + B_8 y^2 + O\left(y^{\frac{3}{2}} e^{-A'\sqrt{\log(y+2)}} \right), \tag{119}$$

where B_8 is an absolute constant whose specific value is immaterial.

The evaluation of the remaining sums $W_i(y)$ has a different origin from that of $W_1(y)$ but still involves the use of multiplicative functions in such a manner that a single summation suffices in each case. First, by (116), (115), and (99),

$$W_2(y) = \frac{1}{2} C_7 \zeta(0) M(0) \sum_{d\Delta \le y} \frac{\psi_2(\Delta d)\, \Theta(\Delta d, 0)\, \mu(\Delta)\, E^{\dagger}(\Delta)\, \tau(\Delta)\, \Gamma^2(d)}{\Delta d \theta_2(\Delta d)}$$

$$= \frac{1}{2} C_7 \zeta(0) M(0) \sum_{n \le y} a'_n, \quad \text{say},$$

the summand a'_n being multiplicative and equal to

$$\frac{\psi_2(n)\, \Theta(n,0)}{n\theta_2(n)} \sum_{d\Delta = n} \mu(\Delta)\, E^{\dagger}(\Delta)\, \tau(\Delta)\, \Gamma^2(d).$$

Owing to (70), a'_n is only non-zero when n is square-free and, by (60), (76), and (78), is equal to

$$\frac{\psi_2(p)\, \Theta(p,0)}{p\theta_2(p)} (\Gamma^2(p) - \tau(p)) = -\frac{1}{p\theta_2^2(p)} \left(1 + \frac{1}{p\theta_2(p)}\right)^{-1} \left(1 + \frac{1}{\theta_2(p)}\right)^{-1}$$

$$= -\frac{1}{p(p-2)\theta_1(p)} \tag{120}$$

when n is an odd prime p; also $a_2 = 1$. Hence

$$W_2(y) = \frac{1}{2} C_7 \zeta(0) M(0) \sum_{n=1}^{\infty} a'_n + O\left(\frac{1}{y^2}\right) \tag{121}$$

$$= \frac{1}{2} C_7 \zeta(0) M(0)(1+1) \prod_{p>2} \left(1 - \frac{1}{p(p-1)\theta_1(p)}\right) + O\left(\frac{1}{y^2}\right)$$

$$= \zeta(0) \prod_{p>2} \frac{(p-2)}{\theta_2(p)} \cdot \frac{\theta_1(p)}{\theta_2(p)} \left(1 - \frac{1}{p}\right) \frac{(p-1)}{(p-2)} \cdot \frac{\theta_2(p)}{\theta_1(p)} + O\left(\frac{1}{y^2}\right)$$

$$= \zeta(0) \prod_{p>2} \frac{(p-1)^2}{p\theta_2(p)} + O\left(\frac{1}{y^2}\right)$$

$$= C_8\,\zeta(0) + O\left(\frac{1}{y^2}\right),$$

by the relation

$$p(p-2)\,\theta_1(p) - 1 = p(p-2)\,\theta_2(p) + p(p-2) - 1 = p(p-1)\,\theta_2(p)$$

and (76), (78), and (27).

Secondly, in considering $W_3(y)$, we are fain to identify the summand in the sum over d that represents $V_3(y/\varDelta, \varDelta)$. Since this is seen to be

$$\frac{1}{d}\,C_7\,\zeta(0)\,M(0)\,\psi_2(\varDelta d)\,\Theta(\varDelta d, 0)\,\Gamma^2(d)\left(-\frac{3}{2} + \log 2 + \frac{\zeta'(0)}{\zeta(0)} + \frac{M'(0)}{M(0)} + \gamma\right)$$

$$+ \frac{1}{d}\,C_7\,\zeta(0)\,M(0)\,\psi_2(\varDelta d)\,\Theta(\varDelta d, 0)\,\Gamma^2(d)\left(-\log d\varDelta + \frac{\Theta'(\varDelta d, 0)}{\Theta(\varDelta d, 0)} + \sum_{\varpi \mid d} \frac{\log \varpi}{\Gamma^2(\varpi)}\right)$$

by another referral to (115) and (99), we infer from (116) that

$$(122)\quad W_3(y) = \left(-\frac{3}{2} + \log 2 + \frac{\zeta'(0)}{\zeta(0)} + \frac{M'(0)}{M(0)} + \gamma\right) C_7\,\zeta(0)\,M(0) \sum_{n \leqq y} a'_n$$
$$+ C_7\,\zeta(0)\,M(0) \sum_{n \leqq y} a'_n \left(-\log n + \frac{\Theta'(n, 0)}{\Theta(n, 0)}\right)$$
$$+ C_7\,\zeta(0)\,M(0) \sum_{n \leqq y} \frac{\psi_2(n)\,\Theta(n, 0)}{n\,\theta_2(n)} \sum_{d\varDelta = n} \mu(\varDelta)\,E^\dagger(\varDelta)\,\tau(\varDelta)\,\Gamma^2(d) \sum_{\varpi \mid d} \frac{\log \varpi}{\Gamma^2(\varpi)}$$
$$= \left(-\frac{3}{2} + \log 2 + \frac{\zeta'(0)}{\zeta(0)} + \frac{M'(0)}{M(0)} + \gamma\right) C_7\,\zeta(0)\,M(0) \sum_{n \leqq y} a'_n$$
$$+ C_7\,\zeta(0)\,M(0) \sum_{n \leqq y} a''_n + C_7\,\zeta(0)\,M(0) \sum_{n \leqq y} a'''_n$$
$$= W_5(y) + W_6(y) + W_7(y), \quad \text{say},$$

where it is immediate that

$$(123)\quad W_5(y) = \left(-3 + 2\log 2 + 2\,\frac{\zeta'(0)}{\zeta(0)} + 2\,\frac{M'(0)}{m(0)} + 2\gamma\right) C_8\,\zeta(0) + O\left(\frac{1}{y^2}\right)$$

from the calculation that gave (121).

To treat the series contained in $W_6(y)$, delimit the range of summation to obtain

$$\sum_{n \leqq y} a_n'' = \sum_{n=1}^{\infty} a_n'' + O\left(\frac{\log 2y}{y^2}\right) \tag{124}$$

and introduce the multiplicative factor

$$R(n,u) = e^{u(-\log n + \Theta'(n,0)/\Theta(n,0))}$$

in order to evaluate the infinite series as the derivative of

$$\sum_{n=1}^{\infty} a_n' R(n,u) = \prod_p \left(1 + a_p' R(p,u)\right)$$

at $u = 0$. Accordingly

$$\begin{aligned}\sum_{n=1}^{\infty} a_n'' &= \left(\sum_{n=1}^{\infty} a_n'\right) \sum_p \frac{a_p'(-\log p + \Theta'(p,0)/\Theta(p,0))}{1 + a_p'} \\ &= \left(\sum_{n=1}^{\infty} a_n'\right) \left\{-\frac{1}{2}\log 2 - \sum_{p>2} \frac{\log p}{p(p-1)\theta_2(p)}\left(-1 + \frac{1}{\theta_1(p)}\right)\right\} \\ &= \left(-\log 2 + 2\sum_{p>2} \frac{\log p}{p(p-1)\theta_2(p)}\right) \prod_{p>2} \frac{(p-1)\theta_2(p)}{(p-2)\theta_1(p)}\end{aligned}$$

and

$$W_6(y) = \left(-\log 2 + 2\sum_{p>2} \frac{\log p}{p(p-1)\theta_2(p)}\right) C_8 \zeta(0) + O\left(\frac{\log 2y}{y^2}\right), \tag{125}$$

whither we are led in turn by (124), (76), and (120).

The estimation of $W_7(y)$ is similar to that of $W_6(y)$ save that we avail ourselves of the multiplicative function

$$e^{u\left(\sum_{p \mid d} \log \varpi / \Gamma^2(\varpi)\right)}$$

to take account of the additive factor appearing in a_n'''. Suppressing therefore the details of the calculation in this instance, we report that

$$\begin{aligned}W_7(y) &= \left(2\sum_{p>2} \frac{\log p}{p(p-1)}\right) C_8 \zeta(0) + O\left(\frac{\log 2y}{y^2}\right) \\ &= \left(-\log 2 + 2\sum_p \frac{\log p}{p(p-1)}\right) C_8 \zeta(0) + O\left(\frac{\log 2y}{y^2}\right),\end{aligned}$$

which equation together with (122), (123), and (125) allows us to conclude that

$$(126)\quad W_3(y) = \left(-3 + 2\frac{\zeta'(0)}{\zeta(0)} + 2\gamma + 2\sum_p \frac{\log p}{p(p-1)}\right) C_8 \zeta(0) + O\left(\frac{\log 2y}{y^2}\right)$$

because

$$\frac{M'(0)}{M(0)} = \sum_{p>2}\left(-\frac{\log p}{\theta_1(p)} + \frac{\log p}{p-1}\right) = -\sum_{p>2}\frac{\log p}{p(p-1)\theta_1(p)}$$

by (76).

As for the remaining constituent $W_4(y)$ in (117), it is evident from (115) and (114) that the above methods yield

$$(127)\qquad W_4(y) = B_9 + O\left(\frac{y^\varepsilon}{y^2}\right)$$

for some absolute constant B_9.

The required formula for $\mathscr{J}(y)$ now follows from (117), (119), (121), (126), and (127), which yield

$$(128)\qquad \mathscr{J}(y) = \frac{1}{3} C_8 \frac{\zeta(2)\zeta(3)}{\zeta(6)} y^3 - \frac{1}{2} C_8 y^2 \log^2 y$$

$$+ C_8 y^2 \log y \left(1 - \frac{\zeta'(0)}{\zeta(0)} - \gamma - \sum_p \frac{\log p}{p(p-1)}\right) + B_{10} y^2 + O\left(y^{\frac{3}{2}} e^{-A'\sqrt{\log(y+2)}}\right)$$

after the value $-\frac{1}{2}$ for $\zeta(0)$ has been inserted into the estimates.

11. The final theorem

The going gets easier as we approach the summit since all we now have to do is to gather together the estimates already found.

First, since

$$C_5 C_6 C_8 = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\left(1 - \frac{1}{p(p-2)}\right)\frac{(p-1)^2}{\theta_2(p)p}$$

$$= \prod_{p>2}\frac{p(p-2)}{(p-1)^2}\cdot\frac{\theta_2(p)}{p-2}\cdot\frac{(p-1)^2}{\theta_2(p)p} = 1$$

by (54), (60), and (103), we infer from (63) and (128) that

$$J_4(x;Q) = \frac{1}{6}\frac{\zeta(2)\zeta(3)}{\zeta(6)}\frac{x^3}{Q} - \frac{1}{4}x^2\log^2\frac{x}{Q} + \frac{1}{2}\left(1 - \frac{\zeta'(0)}{\zeta(0)} - \gamma - \sum_p \frac{\log p}{p(p-1)}\right) x^2 \log\frac{x}{Q}$$

$$+ B_{10} x^2 + O\left(Q^{\frac{1}{2}} x^{\frac{3}{2}} e^{-A'\sqrt{\log\{(x/Q)+2\}}}\right) + O\left(\frac{x^2}{\log^A x}\right),$$

wherefore we gain

$$S_4^*(x;Q,Q_1) = \frac{1}{6}\frac{\zeta(2)\zeta(3)}{\zeta(6)}x^3\left(\frac{1}{Q_1}-\frac{1}{Q}\right) - \frac{1}{2}x^2\log x\log\frac{Q}{Q_1} - \frac{1}{4}x^2(\log^2 Q_1 - \log^2 Q)$$

$$+\frac{1}{2}\left(1-\frac{\zeta'(0)}{\zeta(0)}-\gamma-\sum_p\frac{\log p}{p(p-1)}\right)x^2\log\frac{Q}{Q_1} + O\left(Q^{\frac{1}{2}}x^{\frac{3}{2}}e^{-A'\sqrt{\log\{(x/Q)+2\}}}\right) + O\left(\frac{x^2}{\log^A x}\right)$$

by (20). Hence, combining this with (17) and (30) in (15), we have

$$S_1^*(x;Q,Q_1) = \frac{1}{6}\frac{\zeta(2)\zeta(3)}{\zeta(6)}x^3\left(\frac{1}{Q_1}-\frac{1}{Q}\right) - 3\left(\frac{\zeta'(0)}{\zeta(0)}+\gamma+\sum_p\frac{\log p}{p(p-1)}\right)x^2\log\frac{Q}{Q_1}$$

$$-\frac{3}{2}x^2(\log^2 Q_1 - \log^2 Q) + \frac{1}{\zeta(2)}Qx\log^2 x + O\left(Qx\log x\log^2\frac{2x}{Q}\right)$$

$$+O\left(Q^{\frac{1}{2}}x^{\frac{3}{2}}e^{-A'\sqrt{\log\{(x/Q)+2\}}}\right) + O\left(\frac{x^2}{\log^A x}\right),$$

which equation partial summation transforms into

$$S_1(x;Q,Q_1) = \frac{\zeta(2)\zeta(3)}{\zeta(6)}x^3\log\frac{Q}{Q_1} - 3\left(\frac{\zeta'(0)}{\zeta(0)}+\gamma+\sum_p\frac{\log p}{p(p-1)}\right)x^2(Q-Q_1)$$

$$+3x^2(Q\log Q - Q - Q_1\log Q_1 + Q_1) + \frac{1}{2\zeta(2)}(Q^2-Q_1^2)x\log^2 x$$

$$+O\left\{x\log x\left(Q^2\log^3\frac{2x}{Q}+Q_1^2\log^3\frac{2x}{Q_1}\right)\right\} + O\left(Q^{\frac{3}{2}}x^{\frac{3}{2}}e^{-A'\sqrt{\log\{(x/Q)+2\}}}\right) + O\left(\frac{x^3}{\log^A x}\right)$$

$$= \frac{\zeta(2)\zeta(3)}{\zeta(6)}x^3\log\frac{Q}{Q_1} + 3Qx^2\log Q - 3\left(1+\frac{\zeta'(0)}{\zeta(0)}+\gamma+\sum_p\frac{\log p}{p(p-1)}\right)Qx^2$$

$$+\frac{1}{2\zeta(2)}Q^2x\log^2 x + O\left(Q^2x\log x\log^2\frac{2x}{Q}\right)$$

$$+O\left(Q^{\frac{3}{2}}x^{\frac{3}{2}}e^{-A'\sqrt{\log\{(x/Q)+2\}}}\right) + O\left(\frac{x^3}{\log^A x}\right)$$

because of (3) and (5).

This, (12), and (10) then yield the estimate

$$S(x,Q) = \frac{1}{2\zeta(2)}Q^2x\log^2 x + O\left(Q^2x\log x\log^2\frac{2x}{Q}\right)$$

$$+O\left(Q^{\frac{3}{2}}x^{\frac{3}{2}}e^{-A'\sqrt{\log\{(x/Q)+2\}}}\right) + O\left(\frac{x^3}{\log^A x}\right)$$

that we need.

At last we have the theorems we sought. First there is

Theorem 1. *Let*

$$S(x, Q) = \sum_{k \leqq Q} \phi(k) \sum_{\substack{0 < a \leqq k \\ (a,k)=1}} E^3(x; a, k),$$

where $E(x; a, k)$ is defined in the Introduction. Then, as $x \to \infty$,

$$S(x, Q) = o\left(Q^{\frac{3}{2}} x^{\frac{3}{2}} \log^{\frac{3}{2}} x\right) + O\left(\frac{x^3}{\log^A x}\right)$$

when $Q = o(x/\log x)$.

We also have

Theorem 2. *If $x/\log x \leqq Q \leqq x$, then*

$$S(x, Q) = \frac{1}{2\zeta(2)} Q^2 x \log^2 x + O\left(Q^2 x \log x \log^2 \frac{2x}{Q}\right).$$

Noting the o-term in Theorem 1 can be improved when $Q < x/\log^{4+\varepsilon} x$, we end by remarking on the linkage between these results and those in VII.

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