Project Euler

Steven Miller (sjm1@williams.edu)

Multiples of 3 and 5

Problem 1

If we list all the natural numbers below 10 that are multiples of 3 or 5, we get 3, 5, 6 and 9. The sum of these multiples is 23.

Find the sum of all the multiples of 3 or 5 below 1000.

```
(*For this problem we could go through and keep track of the multiples of
 3 and 5 and add. We can also use the formula that the sum of the first
 n integers is n(n+1)/2. We use this and inclusion/exclusion to quickly
 get a formula for the sum of all multiples of 3 and 5 up to 1000.*)
(*We can't use N as a variable in Mathematica as that means numeric,
so let's use n. Let sum(n) mean the sum of all integers up to
 n. The sum of all multiples of 3 or 5 below 1000 is the following:*)
(* 3 sum(n/3) + 5 sum(n/5) - 15 sum(n/15) *)
(* in the arguments below we use the floor function to make sure we're
 evaluating the sums at the largest integer at most a given argument. *)
problem1sum[n_] := Floor[n] Floor[n + 1] / 2;
problem1answer[n_] :=
 Simplify[3 problem1sum[n / 3] + 5 problem1sum[n / 5] - 15 problem1sum[n / 15]]
problem1answer[999]
(* Here is the brute force answer *)
Sum[ If[Mod[n, 3] == 0 || Mod[n, 5] == 0, n, 0], {n, 1, 999}]
(* this is a good way to check and is a bit easier to code *)
(* I originally got this wrong as I forgot it said BELOW and included 1000 *)
```

```
233 168
```

```
233 168
```

Even Fibonacci numbers

Problem 2

Each new term in the Fibonacci sequence is generated by adding the previous two terms. By starting with 1 and 2, the first 10 terms will be:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

By considering the terms in the Fibonacci sequence whose values do not exceed four million, find the sum of the even-valued terms.

```
(* one has to be careful;
Mathematica has a predefined command for the Fibonaccis,
but they have F1 = 1 and F2 = 1; so you might want to shift by 1. *)
(* the fibonacci numbers are 1, 2, 3, 5, 8, 13, 21, 34, ... *)
(* mod 2 they are 1, 0, 1, 1, 0, 1, 1, 0, ... *)
(* so we only need the numbers whose index is 2 mod 3 *)
(* this program uses the Fibonacci formula and adds every third *)
(* it does not require us to know ahead of
 time what the largest Fibonacci is under 4000000 *)
(* the argument below starts the sum at zero, starts our current at F[2],
and then if the current value is less than 4000000 addds to our sum *)
(* note this uses the fact that we have a hard-
 coded formula for the Fibonaccis *)
(* if we don't want to use that we can just walk down by using the recurrence *)
Fib[n_] := Fibonacci[n + 1];
n = 2;
sum = 0;
current = Fib[2];
While[current < 4 000 000,
  {
   sum = sum + current;
   n = n + 3;
   current = Fib[n];
  }];
Print[sum]
```

4 613 732

```
(* this coe plays with things a bit so we
 can just experimentally find when we hit 4000000 *)
(* we then sum every third *)
Fib[32]
Sum[Fib[2 + 3 * m], {m, 0, 10}]
```

3 524 578

4 613 732

```
(* we can also use Binet's formula and
 the fact that we have two geometric series *)
(* we have Fibonacci[n] = (1/Sqrt[5]) Phi^n - (1/Sqrt[5]) (1-Phi)^n *)
(* remembering to shift *)
(* could solve with geometric series formula *)
(* assuming we know how far to go, we also get the right answer *)
Phi = (1 + Sqrt[5]) / 2;
phi = 1 - Phi;
Simplify[(Phi / Sqrt[5]) Phi^2 ((1 - (Phi^3)^11) / (1 - Phi^3)) -
  (phi / Sqrt[5]) phi^2 ((1 - (phi^3)^11) / (1 - phi^3))]
```

4 613 732

```
(* Another possibility is to find a recurrence for just the evens:
   a(n)=a(n-1)+a(n-2)
        =2 a(n-2)+a(n-3)
        =3 a(n-3)+2 a(n-4)
        =5 a(n-3)-2 a(n-5)
        =5 a(n-3)-2 a(n-4)+2 a(n-6)
        =5 a(n-3)-a(n-4)-a(n-5)+a(n-6)
        =4 a(n-3)+a(n-6)
          this works and allows us to bypass mod conditions! *)
```

Largest prime factor

Problem 3

The prime factors of 13195 are 5, 7, 13 and 29.

What is the largest prime factor of the number 600851475143 ?

```
(* Mathematica has a factorization algorith so this could be very easy! *)
FactorInteger[600 851 475 143]
```

{{71, 1}, {839, 1}, {1471, 1}, {6857, 1}}

```
(* A longer version is to use a list of primes *)
(* if you don't want to use a list of primes you can just divide by integers *)
(* for large numbers this will have problems,
or numbers with a large prime factor *)
(* note we have to be careful about how often a prime divides a number *)
largest = 1;
current = 600 851 475 143;
start = current;
current = 64 * 3;
n = 1;
p = Prime[n];
While[current > 1 && current ≤ Sqrt[start],
  {
   If[Mod[current, p] == 0,
     {
      current = current / p;
      largest = p;
     },
     {
      n = n + 1;
      p = Prime[n];
     }];
  }];
Print[largest];
```

3

Largest palindrome product

Problem 4

A palindromic number reads the same both ways. The largest palindrome made from the product of two 2-digit numbers is 9009 = 91 × 99.

Find the largest palindrome made from the product of two 3-digit numbers.

```
(* lots of ways to do this one, first one that made me stop for a bit *)
(* we could factor palindromic numbers and see which work *)
(* or we could multiply three digit numbers *)
(* we need a good test for palindromic *)
(* we can write a simple program that reverses the digits of a number *)
(* mathematica has a function that would give us the digits as a list,
but fun to write a simple program to do this ourselves using modular arithmetic *)
```

```
reverse[num_] := Module[{},
   digits = Ceiling[Log[10, num]];
   newnum = 0;
   temp = num;
   For[d = 1, d ≤  digits, d++,
    {
     x = Mod[temp, 10];
     newnum = newnum + x * 10 ^ (digits - d);
     temp = (temp - x ) / 10;
    }];
   (*Print[newnum];*)
   Return[newnum];
  ];
palindromecheck[n_] := If[n - reverse[n] == 0, 1, 0];

max = 0;
Timing[
 For[a = 999, a ≥ 100, a--,
   For[b = a, b ≥ 100, b--,
     {
      curr = a * b;
      If[palindromecheck[curr] == 1 && curr > max, max = curr];
      If[a b < max, b = 0]; (* exits loop *)
     }];
  ];
]
Print[max];

(*largest = 1;
pair = {1,1};
For[x1 = 900, x1 ≤ 999, x1++,
 For[x2 = 900, x2 ≤ 999, x2++,
  {
   prod = x1 x2;
   If[prod > largest,
    If[palindromecheck[prod] == 1,
      {
       largest = prod;
       pair = {x1,x2};
      }];
   ];
  }]];
```

```
Print[largest,  " ", pair]*)
```

```
{1.294808, Null}
```

```
906 609
```

```
reverse[1034]
palindromecheck[9332]
```

```
4301
```

Smallest multiple

Problem 5

2520 is the smallest number that can be divided by each of the numbers from 1 to 10 without any remainder.

What is the smallest positive number that is evenly divisible by all of the numbers from 1 to 20?

The way to solve this is to note that the least common multiple is the product of all primes at most 20, where each prime is taken to the largest power so that it is still less than 20. Thus for the prime p, its power is r_p = log[p,20].

```
lcm[num_] := Product[Prime[n]^Floor[Log[Prime[n], 1.0 num]], {n, 1, PrimePi[num]}];
lcm[10]
lcm[20]
```

```
2520
```

```
232 792 560
```

Sum square difference

Problem 6

The sum of the squares of the first ten natural numbers is,

$$1^2 + 2^2 + \ldots + 10^2 = 385$$

The square of the sum of the first ten natural numbers is,

$$(1 + 2 + \ldots + 10)^2 = 55^2 = 3025$$

Hence the difference between the sum of the squares of the first ten natural numbers and the square of the sur

Find the difference between the sum of the squares of the first one hundred natural numbers and the square of

The sum of squares up to n is n(n+1)(2n-1)/6, while the square of the sum is (n(n+1)/2)^2. Thus the answer is

```
diffsquaring[n_] := (n (n + 1) / 2) ^2 - n (n + 1) (2 n + 1) / 6
diffsquaring[10]
diffsquaring[100]
```

```
2640
```

```
25 164 150
```

10001st prime

Problem 7

By listing the first six prime numbers: 2, 3, 5, 7, 11, and 13, we can see that the 6th prime is 13.

What is the 10 001st prime number?

We should do this by the Sieve of Erasthones or some such, striking out the numbers that are not prime from our list. We can also use pre-defined functions (cheating, I know!)

```
Prime[6]
Prime[10 001]
```

```
13
```

```
104 743
```

Largest product in a series

Problem 8

The four adjacent digits in the 1000-digit number that have the greatest product are $9 \times 9 \times 8 \times 9 = 5832$.

```
73167176531330624919225119674426574742355349194934
96983520312774506326239578318016984801869478851843
85861560789112949495459501737958331952853208805511
12540698747158523863050715693290963295227443043557
66896648950445244523161731856403098711121722383113
62229893423380308135336276614282806444486645238749
30358907296290491560440772390713810515859307960866
70172427121883998797908792274921901699720888093776
65727333001053367881220235421809751254540594752243
52584907711670556013604839586446706324415722155397
53697817977846174064955149290862569321978468622482
83972241375657056057490261407972968652414535100474
82166370484403199890008895243450658541227588666881
16427171479924442928230863465674813919123162824586
17866458359124566529476545682848912883142607690042
24219022671055626321111109370544217506941658960408
07198403850962455444362981230987879927244284909188
84580156166097919133875499200524063689912560717606
05886116467109405077541002256983155200055935729725
71636269561882670428252483600823257530420752963450
```

Find the thirteen adjacent digits in the 1000-digit number that have the greatest product. What is the value of t

```
bignum =
  7 316 717 653 133 062 491 922 511 967 442 657 474 235 534 919 493 496 983 520 312 774 506 326 \
   239 578 318 016 984 801 869 478 851 843 858 615 607 891 129 494 954 595 017 379 583 319 528 \
   532 088 055 111 254 069 874 715 852 386 305 071 569 329 096 329 522 744 304 355 766 896 648 \
   950 445 244 523 161 731 856 403 098 711 121 722 383 113 622 298 934 233 803 081 353 362 766 \
   142 828 064 444 866 452 387 493 035 890 729 629 049 156 044 077 239 071 381 051 585 930 796 \
   086 670 172 427 121 883 998 797 908 792 274 921 901 699 720 888 093 776 657 273 330 010 533 \
   678 812 202 354 218 097 512 545 405 947 522 435 258 490 771 167 055 601 360 483 958 644 670 \
   632 441 572 215 539 753 697 817 977 846 174 064 955 149 290 862 569 321 978 468 622 482 839 \
   722 413 756 570 560 574 902 614 079 729 686 524 145 351 004 748 216 637 048 440 319 989 000 \
   889 524 345 065 854 122 758 866 688 116 427 171 479 924 442 928 230 863 465 674 813 919 123 \
   162 824 586 178 664 583 591 245 665 294 765 456 828 489 128 831 426 076 900 422 421 902 267 \
   105 562 632 111 110 937 054 421 750 694 165 896 040 807 198 403 850 962 455 444 362 981 230 \
   987 879 927 244 284 909 188 845 801 561 660 979 191 338 754 992 005 240 636 899 125 607 176 \
   060 588 611 646 710 940 507 754 100 225 698 315 520 005 593 572 972 571 636 269 561 882 670 \
   428 252 483 600 823 257 530 420 752 963 450;
bignumlist = IntegerDigits[bignum];
prod = Product[bignumlist[[n]], {n, 1, 13}];
max = prod;
For[n = 14, n ≤ Length[bignumlist] - 12, n++,
  {
   If[prod > 0, prod = prod * bignumlist[[n]] / bignumlist[[n - 13]]];
   If[prod > max, max = prod];
   If[prod == 0,
    {
     While[prod == 0 && n ≤ Length[bignumlist] - 13,
       {
        n = n + 1;
        prod = Product[bignumlist[[k]], {k, n - 12, n}];
       }
      ];
    }];
  }];
Print[max]
```

23 514 624 000

Special Pythagorean triplet

Problem 9

A Pythagorean triplet is a set of three natural numbers, $a < b < c$, for which,

$$a^2 + b^2 = c^2$$

For example, $3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

There exists exactly one Pythagorean triplet for which $a + b + c = 1000$.
Find the product abc.

```
For[a = 1, a ≤ 1000 / Sqrt[2], a++,
  For[b = a, b ≤ (1000 - a) / 2, bn = 1++,
   {
    c = 1000 - a - b;
    If[a^2 + b^2 - c^2 == 0, Print[a b c]];
   }]
 ];
```

31 875 000

Summation of primes

Problem 10

The sum of the primes below 10 is 2 + 3 + 5 + 7 = 17.

Find the sum of all the primes below two million.

```
sum = 0;
prime = 2;
While[prime < 2 000 000,
  {
   sum = sum + prime;
   prime = NextPrime[prime];
  }];
Print[sum]
```

142 913 828 922

Largest product in a grid

Problem 11

In the 20×20 grid below, four numbers along a diagonal line have been marked in red.

```
08 02 22 97 38 15 00 40 00 75 04 05 07 78 52 12 50 77 91 08
49 49 99 40 17 81 18 57 60 87 17 40 98 43 69 48 04 56 62 00
81 49 31 73 55 79 14 29 93 71 40 67 53 88 30 03 49 13 36 65
52 70 95 23 04 60 11 42 69 24 68 56 01 32 56 71 37 02 36 91
22 31 16 71 51 67 63 89 41 92 36 54 22 40 40 28 66 33 13 80
24 47 32 60 99 03 45 02 44 75 33 53 78 36 84 20 35 17 12 50
32 98 81 28 64 23 67 10 26 38 40 67 59 54 70 66 18 38 64 70
67 26 20 68 02 62 12 20 95 63 94 39 63 08 40 91 66 49 94 21
24 55 58 05 66 73 99 26 97 17 78 78 96 83 14 88 34 89 63 72
21 36 23 09 75 00 76 44 20 45 35 14 00 61 33 97 34 31 33 95
78 17 53 28 22 75 31 67 15 94 03 80 04 62 16 14 09 53 56 92
16 39 05 42 96 35 31 47 55 58 88 24 00 17 54 24 36 29 85 57
86 56 00 48 35 71 89 07 05 44 44 37 44 60 21 58 51 54 17 58
19 80 81 68 05 94 47 69 28 73 92 13 86 52 17 77 04 89 55 40
04 52 08 83 97 35 99 16 07 97 57 32 16 26 26 79 33 27 98 66
88 36 68 87 57 62 20 72 03 46 33 67 46 55 12 32 63 93 53 69
04 42 16 73 38 25 39 11 24 94 72 18 08 46 29 32 40 62 76 36
20 69 36 41 72 30 23 88 34 62 99 69 82 67 59 85 74 04 36 16
20 73 35 29 78 31 90 01 74 31 49 71 48 86 81 16 23 57 05 54
01 70 54 71 83 51 54 69 16 92 33 48 61 43 52 01 89 19 67 48
```

The product of these numbers is 26 × 63 × 78 × 14 = 1788696.

What is the greatest product of four adjacent numbers in the same direction (up, down, left, right, or diagonally) in the 20×20 grid?

```
list = {{08, 02, 22, 97, 38, 15, 00, 40, 00, 75, 04, 05, 07, 78, 52, 12, 50, 77, 91, 08},
   {49, 49, 99, 40, 17, 81, 18, 57, 60, 87, 17, 40, 98, 43, 69, 48, 04, 56, 62, 00},
   {81, 49, 31, 73, 55, 79, 14, 29, 93, 71, 40, 67, 53, 88, 30, 03, 49, 13, 36, 65},
   {52, 70, 95, 23, 04, 60, 11, 42, 69, 24, 68, 56, 01, 32, 56, 71, 37, 02, 36, 91},
   {22, 31, 16, 71, 51, 67, 63, 89, 41, 92, 36, 54, 22, 40, 40, 28, 66, 33, 13, 80},
   {24, 47, 32, 60, 99, 03, 45, 02, 44, 75, 33, 53, 78, 36, 84, 20, 35, 17, 12, 50},
   {32, 98, 81, 28, 64, 23, 67, 10, 26, 38, 40, 67, 59, 54, 70, 66, 18, 38, 64, 70},
   {67, 26, 20, 68, 02, 62, 12, 20, 95, 63, 94, 39, 63, 08, 40, 91, 66, 49, 94, 21},
   {24, 55, 58, 05, 66, 73, 99, 26, 97, 17, 78, 78, 96, 83, 14, 88, 34, 89, 63, 72},
   {21, 36, 23, 09, 75, 00, 76, 44, 20, 45, 35, 14, 00, 61, 33, 97, 34, 31, 33, 95},
   {78, 17, 53, 28, 22, 75, 31, 67, 15, 94, 03, 80, 04, 62, 16, 14, 09, 53, 56, 92},
   {16, 39, 05, 42, 96, 35, 31, 47, 55, 58, 88, 24, 00, 17, 54, 24, 36, 29, 85, 57},
   {86, 56, 00, 48, 35, 71, 89, 07, 05, 44, 44, 37, 44, 60, 21, 58, 51, 54, 17, 58},
   {19, 80, 81, 68, 05, 94, 47, 69, 28, 73, 92, 13, 86, 52, 17, 77, 04, 89, 55, 40},
   {04, 52, 08, 83, 97, 35, 99, 16, 07, 97, 57, 32, 16, 26, 26, 79, 33, 27, 98, 66},
   {88, 36, 68, 87, 57, 62, 20, 72, 03, 46, 33, 67, 46, 55, 12, 32, 63, 93, 53, 69},
   {04, 42, 16, 73, 38, 25, 39, 11, 24, 94, 72, 18, 08, 46, 29, 32, 40, 62, 76, 36},
   {20, 69, 36, 41, 72, 30, 23, 88, 34, 62, 99, 69, 82, 67, 59, 85, 74, 04, 36, 16},
   {20, 73, 35, 29, 78, 31, 90, 01, 74, 31, 49, 71, 48, 86, 81, 16, 23, 57, 05, 54},
   {01, 70, 54, 71, 83, 51, 54, 69, 16, 92, 33, 48, 61, 43, 52, 01, 89, 19, 67, 48}};
maxprod = 0;
(* goes through horizontally, vertically and diagonally *)
(* as matrix is so small just do all products if inside *)

For[i = 1, i ≤ 20, i++,
  For[j = 1, j ≤ 20, j++,
   {
    If[j + 3 ≤ 20,
```

```
       {
        tempprod =  Product[list[[i, j + k]], {k, 0, 3}];
        If[ tempprod > maxprod, maxprod = tempprod];
       }];
      If[i + 3 ≤ 20,
       {
        tempprod =  Product[list[[i + k, j]], {k, 0, 3}];
        If[ tempprod > maxprod, maxprod = tempprod];
       }];
      If[i + 3 ≤ 20 && j + 3 ≤ 20,
       {
        tempprod =  Product[list[[i + k, j + k]], {k, 0, 3}];
        If[ tempprod > maxprod, maxprod = tempprod];
       }];
      If[i + 3 ≤ 20 && j - 3 ≥ 1,
       {
        tempprod =  Product[list[[i + k, j - k]], {k, 0, 3}];
        If[ tempprod > maxprod, maxprod = tempprod];
       }];
     }]];
Print[maxprod]
```

70 600 674

Problem 12: Highly divisible triangular number

The sequence of triangle numbers is generated by adding the natural numbers. So the 7th triangle number would be 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28. The first ten terms would be:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

Let us list the factors of the first seven triangle numbers:

```
1: 1
3: 1,3
6: 1,2,3,6
```

```
10: 1,2,5,10
15: 1,3,5,15
21: 1,3,7,21
28: 1,2,4,7,14,28
```

We can see that 28 is the first triangle number to have over five divisors.

What is the value of the first triangle number to have over five hundred divisors?

```
(*TriangleNumber[n_]:=Sum[i,{i,1,n}];*)
trianglenumber[n_] := n (n + 1) / 2;
found = 0;
n = 1;
While[found == 0,
{
   number = trianglenumber[n];
   numdivisors = Length[Divisors[number]];
   If[numdivisors ≥ 501,
    {
     Print["Number is ", number, ""];
     Print["Number of divisors is ", numdivisors];
     found = 1;
    }];
   n = n + 1;
  };
]
```

```
Number is 76 576 500
Number of divisors is 576
```

Large sum

Problem 13

Work out the first ten digits of the sum of the following one-hundred 50-digit numbers.

```
problem13list = {37 107 287 533 902 102 798 797 998 220 837 590 246 510 135 740 250,
46 376 937 677 490 009 712 648 124 896 970 078 050 417 018 260 538,
74 324 986 199 524 741 059 474 233 309 513 058 123 726 617 309 629,
91 942 213 363 574 161 572 522 430 563 301 811 072 406 154 908 250,
23 067 588 207 539 346 171 171 980 310 421 047 513 778 063 246 676,
89 261 670 696 623 633 820 136 378 418 383 684 178 734 361 726 757,
28 112 879 812 849 979 408 065 481 931 592 621 691 275 889 832 738,
44 274 228 917 432 520 321 923 589 422 876 796 487 670 272 189 318,
47 451 445 736 001 306 439 091 167 216 856 844 588 711 603 153 276,
70 386 486 105 843 025 439 939 619 828 917 593 665 686 757 934 951,
62 176 457 141 856 560 629 502 157 223 196 586 755 079 324 193 331,
64 906 352 462 741 904 929 101 432 445 813 822 663 347 944 758 178,
92 575 867 718 337 217 661 963 751 590 579 239 728 245 598 838 407,
58 203 565 325 359 399 008 402 633 568 948 830 189 458 628 227 828,
80 181 199 384 826 282 014 278 194 139 940 567 587 151 170 094 390,
35 398 664 372 827 112 653 829 987 240 784 473 053 190 104 293 586,
```

86 515 506 006 295 864 861 532 075 273 371 959 191 420 517 255 829,
71 693 888 707 715 466 499 115 593 487 603 532 921 714 970 056 938,
54 370 070 576 826 684 624 621 495 650 076 471 787 294 438 377 604,
53 282 654 108 756 828 443 191 190 634 694 037 855 217 779 295 145,
36 123 272 525 000 296 071 075 082 563 815 656 710 885 258 350 721,
45 876 576 172 410 976 447 339 110 607 218 265 236 877 223 636 045,
17 423 706 905 851 860 660 448 207 621 209 813 287 860 733 969 412,
81 142 660 418 086 830 619 328 460 811 191 061 556 940 512 689 692,
51 934 325 451 728 388 641 918 047 049 293 215 058 642 563 049 483,
62 467 221 648 435 076 201 727 918 039 944 693 004 732 956 340 691,
15 732 444 386 908 125 794 514 089 057 706 229 429 197 107 928 209,
55 037 687 525 678 773 091 862 540 744 969 844 508 330 393 682 126,
18 336 384 825 330 154 686 196 124 348 767 681 297 534 375 946 515,
80 386 287 592 878 490 201 521 685 554 828 717 201 219 257 766 954,
78 182 833 757 993 103 614 740 356 856 449 095 527 097 864 797 581,
16 726 320 100 436 897 842 553 539 920 931 837 441 497 806 860 984,
48 403 098 129 077 791 799 088 218 795 327 364 475 675 590 848 030,
87 086 987 551 392 711 854 517 078 544 161 852 424 320 693 150 332,
59 959 406 895 756 536 782 107 074 926 966 537 676 326 235 447 210,
69 793 950 679 652 694 742 597 709 739 166 693 763 042 633 987 085,
41 052 684 708 299 085 211 399 427 365 734 116 182 760 315 001 271,
65 378 607 361 501 080 857 009 149 939 512 557 028 198 746 004 375,
35 829 035 317 434 717 326 932 123 578 154 982 629 742 552 737 307,
94 953 759 765 105 305 946 966 067 683 156 574 377 167 401 875 275,
88 902 802 571 733 229 619 176 668 713 819 931 811 048 770 190 271,
25 267 680 276 078 003 013 678 680 992 525 463 401 061 632 866 526,
36 270 218 540 497 705 585 629 946 580 636 237 993 140 746 255 962,
24 074 486 908 231 174 977 792 365 466 257 246 923 322 810 917 141,
91 430 288 197 103 288 597 806 669 760 892 938 638 285 025 333 403,
34 413 065 578 016 127 815 921 815 005 561 868 836 468 420 090 470,
23 053 081 172 816 430 487 623 791 969 842 487 255 036 638 784 583,
11 487 696 932 154 902 810 424 020 138 335 124 462 181 441 773 470,
63 783 299 490 636 259 666 498 587 618 221 225 225 512 486 764 533,
67 720 186 971 698 544 312 419 572 409 913 959 008 952 310 058 822,
95 548 255 300 263 520 781 532 296 796 249 481 641 953 868 218 774,
76 085 327 132 285 723 110 424 803 456 124 867 697 064 507 995 236,
37 774 242 535 411 291 684 276 865 538 926 205 024 910 326 572 967,
23 701 913 275 725 675 285 653 248 258 265 463 092 207 058 596 522,
29 798 860 272 258 331 913 126 375 147 341 994 889 534 765 745 501,
18 495 701 454 879 288 984 856 827 726 077 713 721 403 798 879 715,
38 298 203 783 031 473 527 721 580 348 144 513 491 373 226 651 381,
34 829 543 829 199 918 180 278 916 522 431 027 392 251 122 869 539,
40 957 953 066 405 232 632 538 044 100 059 654 939 159 879 593 635,

```
29 746 152 185 502 371 307 642 255 121 183 693 803 580 388 584 903,
41 698 116 222 072 977 186 158 236 678 424 689 157 993 532 961 922,
62 467 957 194 401 269 043 877 107 275 048 102 390 895 523 597 457,
23 189 706 772 547 915 061 505 504 953 922 979 530 901 129 967 519,
86 188 088 225 875 314 529 584 099 251 203 829 009 407 770 775 672,
11 306 739 708 304 724 483 816 533 873 502 340 845 647 058 077 308,
82 959 174 767 140 363 198 008 187 129 011 875 491 310 547 126 581,
97 623 331 044 818 386 269 515 456 334 926 366 572 897 563 400 500,
42 846 280 183 517 070 527 831 839 425 882 145 521 227 251 250 327,
55 121 603 546 981 200 581 762 165 212 827 652 751 691 296 897 789,
32 238 195 734 329 339 946 437 501 907 836 945 765 883 352 399 886,
75 506 164 965 184 775 180 738 168 837 861 091 527 357 929 701 337,
62 177 842 752 192 623 401 942 399 639 168 044 983 993 173 312 731,
32 924 185 707 147 349 566 916 674 687 634 660 915 035 914 677 504,
99 518 671 430 235 219 628 894 890 102 423 325 116 913 619 626 622,
73 267 460 800 591 547 471 830 798 392 868 535 206 946 944 540 724,
76 841 822 524 674 417 161 514 036 427 982 273 348 055 556 214 818,
97 142 617 910 342 598 647 204 516 893 989 422 179 826 088 076 852,
87 783 646 182 799 346 313 767 754 307 809 363 333 018 982 642 090,
10 848 802 521 674 670 883 215 120 185 883 543 223 812 876 952 786,
71 329 612 474 782 464 538 636 993 009 049 310 363 619 763 878 039,
62 184 073 572 399 794 223 406 235 393 808 339 651 327 408 011 116,
66 627 891 981 488 087 797 941 876 876 144 230 030 984 490 851 411,
60 661 826 293 682 836 764 744 779 239 180 335 110 989 069 790 714,
85 786 944 089 552 990 653 640 447 425 576 083 659 976 645 795 096,
66 024 396 409 905 389 607 120 198 219 976 047 599 490 197 230 297,
64 913 982 680 032 973 156 037 120 041 377 903 785 566 085 089 252,
16 730 939 319 872 750 275 468 906 903 707 539 413 042 652 315 011,
94 809 377 245 048 795 150 954 100 921 645 863 754 710 598 436 791,
78 639 167 021 187 492 431 995 700 641 917 969 777 599 028 300 699,
15 368 713 711 936 614 952 811 305 876 380 278 410 754 449 733 078,
40 789 923 115 535 562 561 142 322 423 255 033 685 442 488 917 353,
44 889 911 501 440 648 020 369 068 063 960 672 322 193 204 149 535,
41 503 128 880 339 536 053 299 340 368 006 977 710 650 566 631 954,
81 234 880 673 210 146 739 058 568 557 934 581 403 627 822 703 280,
82 616 570 773 948 327 592 232 845 941 706 525 094 512 325 230 608,
22 918 802 058 777 319 719 839 450 180 888 072 429 661 980 811 197,
77 158 542 502 016 545 090 413 245 809 786 882 778 948 721 859 617,
72 107 838 435 069 186 155 435 662 884 062 257 473 692 284 509 516,
20 849 603 980 134 001 723 930 671 666 823 555 245 252 804 609 722,
53 503 534 226 472 524 250 874 054 075 591 789 781 264 330 331 690};
Sum[problem13list[[n]], {n, 1, Length[problem13list]}]
```

5 537 376 230 390 876 637 302 048 746 832 985 971 773 659 831 892 672

Problem 14: Longest Collatz sequence

The following iterative sequence is defined for the set of positive integers:

$n \to n/2$ (n is even)
$n \to 3n + 1$ (n is odd)

Using the rule above and starting with 13, we generate the following sequence:

$$13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$$

It can be seen that this sequence (starting at 13 and finishing at 1) contains 10 terms. Although it has not been proved y
thought that all starting numbers finish at 1.

Which starting number, under one million, produces the longest chain?

NOTE: Once the chain starts the terms are allowed to go above one million.

```
(*  debugging code *)
start = 111;
now = start;
f[x_] := If[Mod[x, 2] == 1, 3 x + 1, x / 2];
list = {};
While[now > 1,
  {
   list = AppendTo[list, now];
   now = f[now];
  }];
list = AppendTo[list, now];
Print[list, " ", Length[list]]
```

```
{111, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438,
  719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051,
  6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244,
  122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1} 70
```

```
max = 1 000 000;
(*safety = 1;*)
maxlength = 0;
maxstart = 0;
For[n = 1, n ≤ max + 5, n++, arraysteps[n] = 0];
Clear[f];
f[x_] := If[Mod[x, 2] == 1, 3 x + 1, x / 2];
arraysteps[1] = 1;
arraysteps[2] = 2;
```

```
(*arraysteps[3] = 8;*)
For[n = 3, n ≤ max, n++,
  {
   temp = {};
   steps = 0;
   curr = n;
   While[curr ≥ n && arraysteps[n] == 0,
    {
     temp = AppendTo[temp, curr];
     steps = steps + 1;
     curr = f[curr];
     (*steps = steps + 1;*)
     (*If[curr >  max && arraysteps[n] == 0,
       {
        Print["Exceed safety * max at ", curr];
        curr = 0;
        n = max + 10;
       }]; (* end of if *)
     *)
    }]; (* end of while *)
   For[j = 1, j ≤ Length[temp], j++,
    {
     x = temp[[j]];
     If[x ≤ max && arraysteps[x] == 0,
      arraysteps[x] = steps + arraysteps[curr] - j + 1];
    }];

   If[ arraysteps[n]  > maxlength,
    {
     maxlength = arraysteps[n];
     maxstart = n;
    }]; (* end of if *)
  }];
(* end of n loop *)
Print["max n is ", maxstart, " and length was ", maxlength, "."];
```

max n is 837 799 and length was 525.

```
arraysteps[7]
```

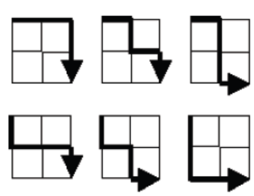
17

```
For[n = 1, n ≤ 20, n++, Print[n, " ", arraysteps[n]]]
```

```
1 1
2 2
3 8
4 3
5 6
6 9
7 17
8 4
9 20
10 7
11 15
12 10
13 10
14 18
15 18
16 5
17 13
18 21
19 21
20 8
```

Problem 15: Lattice paths

Starting in the top left corner of a 2×2 grid, and only being able to move to the right and down, there are exactly 6 routes to the bottom right corner.



How many such routes are there through a 20×20 grid?

```
(* we have a 20 by 20 grid *)
(* we have 20 downs and 20 rights *)
(* answer is (40 choose 20) *)
Binomial[40, 20] Binomial[20, 20]
path[n_] := Binomial[2 n, n]
path[1]
path[2]
path[3]
```

```
137 846 528 820

2

6

20
```

```
(* we can do by a recurrence relation *)
(* if a(i,j) is how many paths from (i,j), so i downs and j rights *)
(* we have a(i,j) = a(i-1,j) + a(i,j-1) *)
(* start with a(1,0) = a(0,1) = 1 *)
(* note a(i,0) = 1 and a(0,i) = 1 *)
(* we would get a(1,1) = a(1,0) + a(0,1) = 2 *)
(* to get to a(2,2) we need a(2,1) and a(1,2) first *)
(* fortunately a(i,j) and a(j,i) are equal *) (* replace down with right... *)
(* so a(2,1) = a(2,0) + a(1,1) *)
(* thus a(2,1) = 1 + 2 = 3 *)
(* thus a(2,2) = a(2,1) + a(1,2) = 6 *)
(* more generally a(i,i) = a(i-1,i) + a(i,i-1) = 2 a(i-1,i) *)
Clear[a]; (* clears a *)

For[i = 1, i ≤ 40, i++,
  For[j  = 1, j ≤ 40, j++, a[i, j] = 0]]; (* initialize all to 0 *)

a[0, 0] = 0; (* these are some initial knowns *)
a[1, 0] = a[0, 1] = 1;
a[1, 1] = 2;
a[2, 0] = a[0, 2] = 1;
a[2, 1] = a[1, 2] = 3;
a[2, 2] = 6;
For[i = 3, i ≤ 20, i++, a[i, 0] = a[0, i] = 1];

(* now we build up our 2-dim recurrence values,
using a(i,j) = a(i-1,j) + a(i,j-1) *)
For[i = 3, i ≤ 20, i++,
  For[j = 1, j ≤ i, j++,
    {
     a[i, j] = a[i - 1, j] + a[i, j - 1]; (* our recurrence *)
     a[j, i] = a[i, j]; (* note must fill in symmetrically *)
    }]; (* end of j loop *)
 ]; (* end of i loop *)

a[3, 2] (* checking some values, answering the problem *)
a[3, 3]
Print["We get for 20 ups and 20 downs ", a[20, 20], "."];
Print["The answer is ", Binomial[40, 20], "."];
```

```
10

20
```

We get for 20 ups and 20 downs 137 846 528 820.

The answer is 137 846 528 820.

Power digit sum

Problem 16

$2^{15} = 32768$ and the sum of its digits is $3 + 2 + 7 + 6 + 8 = 26$.

What is the sum of the digits of the number 2^{1000}?

```
num = 2^1000;
numstring = IntegerDigits[num];
Sum[numstring[[n]], {n, 1, Length[numstring]}]
```

1366

Counting Sundays

Problem 19

You are given the following information, but you may prefer to do some research for yourself.

- 1 Jan 1900 was a Monday.
- Thirty days has September,
  April, June and November.
  All the rest have thirty-one,
  Saving February alone,
  Which has twenty-eight, rain or shine.
  And on leap years, twenty-nine.
- A leap year occurs on any year evenly divisible by 4, but not on a century unless it is divisible by 400.

How many Sundays fell on the first of the month during the twentieth century (1 Jan 1901 to 31 Dec 2000)?

Solution by my student Elliot Chester

```
Timing[sundays = 0(*We start our running total at 0*); start = 2(*January 1,
 1901 was a Tuesday, which is 2 mod 7. We call Sunday 0 mod 7.*);
 (*A loop for each month over our 100 year period*)
 For[which = 1, which < 1200, which++,
   (*The 31 day months, which push the start of the next month back three days
    (ie January/Monday --> February/Thursday. So if one of these months starts *)
   If[Mod[which, 12] == 1 || Mod[which, 12] == 3 || Mod[which, 12] == 5 ||
     Mod[which, 12] == 7 || Mod[which, 12] == 8 || Mod[which, 12] == 10 ||
     Mod[which, 12] == 0, start = Mod[start + 3, 7]];
   (*The 30 day months, which take general form June/Monday --> July/Wednesday*)
   If[ Mod[which, 12] == 4 || Mod[which, 12] == 6 ||
     Mod[which, 12] == 9 || Mod[which, 12] == 11, start = Mod[start + 2, 7]];
   (*February in leap years,where February/Monday -->
    March/Tuesday. Leap Februaries are the 38th month in a 48 (4-month) cycle.
      Normal Februaries are exactly 4 weeks long,
   only Februaries that start on a Sunday add another Sunday
    NOTE: This code does NOT fully account for the century leap year rule*)
   If[Mod[which, 12] == 2 && Mod[which, 48] == 38, start = Mod[start + 1, 7]];
   (*This line says,
   "If a month starts on a Sunday, add a Sunday to our running total*)
   If[start == 0, sundays += 1];
  ]
  Print[sundays]]
```

```
171
```

$\{0.031200, \text{Null}^2\}$

```
(*A solution from the message boards that is a lot neater,
  since apparently Mathematica knows about dates. That's cheating, though :)
 Besides, mine runs faster, even if as noted it is not
 generalizable to arbitrary year ranges quite yet*)
Timing[interval = With[{first = {1901, 1, 1}, last = {2000, 12, 31}},
   DateRange[first, last, "Month"]];
Tally[DayName /@ interval]]
```

Factorial digit sum

Problem 20

$n!$ means $n \times (n - 1) \times \ldots \times 3 \times 2 \times 1$

For example, $10! = 10 \times 9 \times \ldots \times 3 \times 2 \times 1 = 3628800$,
and the sum of the digits in the number 10! is $3 + 6 + 2 + 8 + 8 + 0 + 0 = 27$.

Find the sum of the digits in the number 100!

```
list = IntegerDigits[100!];
Sum[list[[n]], {n, 1, Length[list]}]
```

```
648
```

Amicable numbers

Problem 21

Let d(n) be defined as the sum of proper divisors of n (numbers less than n which divide evenly into n).
If d(a) = b and d(b) = a, where $a \neq b$, then a and b are an amicable pair and each of a and b are called amicable numbers.

For example, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; therefore d(220) = 284. The proper divisors of 284 are 1, 2, 4, 71 and 142; so d(284) = 220.

Evaluate the sum of all the amicable numbers under 10000.

```
sum = 0;
For[a = 1, a ≤ 10 000, a++,
  {
   b = DivisorSigma[1, a] - a;
   divb = DivisorSigma[1, b] - b;
   If[divb == a && a < b && b < 10 000, sum = sum + a + b;];
  }];
Print[sum];
```

```
31 626
```

```
DivisorSigma[1, 15] - 15
```

```
9
```

Non-abundant sums

Problem 23

A perfect number is a number for which the sum of its proper divisors is exactly equal to the number. For example, the sum of the proper divisors of 28 would be 1 + 2 + 4 + 7 + 14 = 28, which means that 28 is a perfect number.

A number n is called deficient if the sum of its proper divisors is less than n and it is called abundant if this sum exceeds n.

As 12 is the smallest abundant number, 1 + 2 + 3 + 4 + 6 = 16, the smallest number that can be written as the sum of two abundant numbers is 24. By mathematical analysis, it can be shown that all integers greater than 28123 can be written as the sum of two abundant numbers. However, this upper limit cannot be reduced any further by analysis even though it is known that the greatest number that cannot be expressed as the sum of two abundant numbers is less than this limit.

Find the sum of all the positive integers which cannot be written as the sum of two abundant numbers.

```
f[n_] := DivisorSigma[1, n] - n
list = {};
For[n = 2, n ≤ 28123, n++, If[f[n] > n, list = AppendTo[list, n]]]
newlist = {};
Print["The length of the list is ", Length[list]];

(*For[n= 1, n ≤ Length[list], n++,
 {
  If[Mod[n,400] == 0,
   Print["We are at ", n, " of ", Length[list], " and found ", Length[newlist]]];
  x = list[[n]];
  For[m = 1, m ≤ n, m++,
   {
    y = list[[m]];
    If[x+y > 28123, m = 2n+ 100,
     If[MemberQ[newlist,x+y] == False, newlist = AppendTo[newlist, x+y]];];
   }];
 }];
Print["Found ", Length[newlist]];
sum = 0;
For[n = 1, n ≤ 28123, n++, If[MemberQ[newlist,n] == False, sum = sum+n]];*)
```

```
The length of the list is 6965
```

```
sum = Sum[i, {i, 1, 23}];
(*  first number sum two abundants is 24 *)
Print["Number abundants is ", Length[list]];
For[n = 25, n ≤ 28123, n++,
  {
   If[Mod[n, 400] == 0,
    Print["We are at ", n, " of 28123 and sum is ", sum, "."]];
   work = 0;
   For[i = 1, i ≤ Length[list], i++.
     {
      x = n - list[[i]];
      If[x < 0, i = 30000,
        {
         If[MemberQ[list, x] == True,
           {
            work = 1;
            i = 30000;
           }];
        }];
     }]; (* end of i loop *)
   If[work == 0, sum = sum + n];
  }];
Print["Sum is ", sum]
```

```
Number abundants is 6965
We are at 400 of 28123 and sum is 40266.
We are at 800 of 28123 and sum is 160266.
We are at 1200 of 28123 and sum is 297850.
We are at 1600 of 28123 and sum is 434888.
We are at 2000 of 28123 and sum is 573153.
We are at 2400 of 28123 and sum is 733582.
We are at 2800 of 28123 and sum is 905682.
We are at 3200 of 28123 and sum is 1088959.
We are at 3600 of 28123 and sum is 1278579.
We are at 4000 of 28123 and sum is 1490743.
We are at 4400 of 28123 and sum is 1712672.
We are at 4800 of 28123 and sum is 1946917.
We are at 5200 of 28123 and sum is 2162602.
We are at 5600 of 28123 and sum is 2371859.
```

```
We are at 6000 of 28123 and sum is 2 621 368.
We are at 6400 of 28123 and sum is 2 795 012.
We are at 6800 of 28123 and sum is 2 992 336.
We are at 7200 of 28123 and sum is 3 125 479.
We are at 7600 of 28123 and sum is 3 228 923.
We are at 8000 of 28123 and sum is 3 353 255.
We are at 8400 of 28123 and sum is 3 443 268.
We are at 8800 of 28123 and sum is 3 528 566.
We are at 9200 of 28123 and sum is 3 645 361.
We are at 9600 of 28123 and sum is 3 701 953.
We are at 10 000 of 28123 and sum is 3 731 004.
We are at 10 400 of 28123 and sum is 3 751 698.
We are at 10 800 of 28123 and sum is 3 783 389.
We are at 11 200 of 28123 and sum is 3 805 333.
We are at 11 600 of 28123 and sum is 3 816 924.
We are at 12 000 of 28123 and sum is 3 828 725.
We are at 12 400 of 28123 and sum is 3 853 099.
We are at 12 800 of 28123 and sum is 3 903 425.
We are at 13 200 of 28123 and sum is 3 929 313.
We are at 13 600 of 28123 and sum is 3 955 919.
We are at 14 000 of 28123 and sum is 3 997 248.
We are at 14 400 of 28123 and sum is 4 039 939.
We are at 14 800 of 28123 and sum is 4 039 939.
We are at 15 200 of 28123 and sum is 4 039 939.
We are at 15 600 of 28123 and sum is 4 070 867.
We are at 16 000 of 28123 and sum is 4 070 867.
We are at 16 400 of 28123 and sum is 4 087 054.
We are at 16 800 of 28123 and sum is 4 087 054.
We are at 17 200 of 28123 and sum is 4 087 054.
We are at 17 600 of 28123 and sum is 4 104 315.
We are at 18 000 of 28123 and sum is 4 122 206.
We are at 18 400 of 28123 and sum is 4 122 206.
We are at 18 800 of 28123 and sum is 4 140 643.
We are at 19 200 of 28123 and sum is 4 159 710.
We are at 19 600 of 28123 and sum is 4 159 710.
We are at 20 000 of 28123 and sum is 4 159 710.
```

```
We are at 20 400 of 28123 and sum is 4 179 871.
We are at 20 800 of 28123 and sum is 4 179 871.
We are at 21 200 of 28123 and sum is 4 179 871.
We are at 21 600 of 28123 and sum is 4 179 871.
We are at 22 000 of 28123 and sum is 4 179 871.
We are at 22 400 of 28123 and sum is 4 179 871.
We are at 22 800 of 28123 and sum is 4 179 871.
We are at 23 200 of 28123 and sum is 4 179 871.
We are at 23 600 of 28123 and sum is 4 179 871.
We are at 24 000 of 28123 and sum is 4 179 871.
We are at 24 400 of 28123 and sum is 4 179 871.
We are at 24 800 of 28123 and sum is 4 179 871.
We are at 25 200 of 28123 and sum is 4 179 871.
We are at 25 600 of 28123 and sum is 4 179 871.
We are at 26 000 of 28123 and sum is 4 179 871.
We are at 26 400 of 28123 and sum is 4 179 871.
We are at 26 800 of 28123 and sum is 4 179 871.
We are at 27 200 of 28123 and sum is 4 179 871.
We are at 27 600 of 28123 and sum is 4 179 871.
We are at 28 000 of 28123 and sum is 4 179 871.
Sum is 4 179 871
```

n

```
25
```

1000-digit Fibonacci number

Problem 25

The Fibonacci sequence is defined by the recurrence relation:

$F_n = F_{n-1} + F_{n-2}$, where $F_1 = 1$ and $F_2 = 1$.

Hence the first 12 terms will be:

$F_1 = 1$
$F_2 = 1$
$F_3 = 2$
$F_4 = 3$
$F_5 = 5$
$F_6 = 8$
$F_7 = 13$
$F_8 = 21$
$F_9 = 34$
$F_{10} = 55$
$F_{11} = 89$
$F_{12} = 144$

The 12th term, F_{12}, is the first term to contain three digits.

What is the first term in the Fibonacci sequence to contain 1000 digits?

```
fib[n_] := (1/Sqrt[5]) ((1 + Sqrt[5])/2)^n - (1/Sqrt[5]) ((1 - Sqrt[5])/2)^n;
fibapprox[n_] := (1/Sqrt[5]) ((1 + Sqrt[5])/2)^n;
napprox = (999 Log[10] - Log[1./Sqrt[5]]) / Log[(1 + Sqrt[5])/2]
```

```
4781.86
```

```
Log[10., Fibonacci[4782]]
```

```
999.029
```

Digit fifth powers

Problem 30

Surprisingly there are only three numbers that can be written as the sum of fourth powers of their digits:

$1634 = 1^4 + 6^4 + 3^4 + 4^4$
$8208 = 8^4 + 2^4 + 0^4 + 8^4$
$9474 = 9^4 + 4^4 + 7^4 + 4^4$

As $1 = 1^4$ is not a sum it is not included.

The sum of these numbers is 1634 + 8208 + 9474 = 19316.

Find the sum of all the numbers that can be written as the sum of fifth powers of their digits.

```
(* there are only finitely many,
and eventually 10^n grows faster than a lot of 9s*)
(* I gambled at first -- I checked
 up to a large number and hoped it found all *)
(* fortunately the only way we can get such a large
 number is to essentially have almost all 9s *)
(* even one digit of 8^5 instead of 9^5 is bad *)
(* seven 9's gives 7 9^5 = 413343 which is too small,
as it is less than 1000000 *)
(* thus we only need to check up to 413343 and that'll speed things up *)

numwork = 0;
runningsum = 0;
For[n = 10, n ≤ 10 000 000, n++,
  {
   temp = IntegerDigits[n];
   If[n == Sum[temp[[i]]^5, {i, 1, Length[temp]}],
    {Print[n]; numwork = numwork + 1; runningsum = runningsum + n;}];
  }];
Print[numwork, " ", runningsum]
```

```
4150
4151
54 748
92 727
93 084
194 979
6 443 839
```

```
8^5
7 × 9^5
```

```
32 768
413 343
```

Coin sums

Problem 31

In England the currency is made up of pound, £, and pence, p, and there are eight coins in general circulation:

1p, 2p, 5p, 10p, 20p, 50p, £1 (100p) and £2 (200p).

It is possible to make £2 in the following way:

1×£1 + 1×50p + 2×20p + 1×5p + 1×2p + 3×1p

How many different ways can £2 be made using any number of coins?

```
S = 200;
num = 1; (* from using the 200 pence coin *)
For[a = 0, a ≤ Floor[S / 100], a++, For[b = 0,
  b ≤ Floor[(S - a * 100) / 50], b++, For[c = 0, c ≤ Floor[(S - a * 100 - b * 50) / 20],
   c++, For[d = 0, d ≤ Floor[(S - a * 100 - b * 50 - c * 20) / 10], d++,
    For[e = 0, e ≤ Floor[(S - a * 100 - b * 50 - c * 20 - d * 10) / 5], e++, For[f = 0,
      f ≤ Floor[(S - a * 100 - b * 50 - c * 20 - d * 10 - e * 5) / 2], f++, num = num + 1]]]]]]
Print[num]
```

73 682

Digit cancelling fractions

Problem 33

The fraction $^{49}/_{98}$ is a curious fraction, as an inexperienced mathematician in attempting to simplify it may incorrectly believe that $^{49}/_{98} = {}^{4}/_{8}$, which is correct, is obtained by cancelling the 9s.

We shall consider fractions like, $^{30}/_{50} = {}^{3}/_{5}$, to be trivial examples.

There are exactly four non-trivial examples of this type of fraction, less than one in value, and containing two digits in the numerator and denominator.

If the product of these four fractions is given in its lowest common terms, find the value of the denominator.

```
prod = 1;
For[a = 1, a ≤ 9, a++,
 For[b = 1, b ≤ 9, b++,
  For[c = 1, c ≤ 9, c++,
   {
    x = (10 a + b) / (10 * b + c);
    y = (10 a + b) / (10 * c + a);
    If[x == a / c && x < 1, prod = prod * a / c];
    If[x ≠ y && y == b / c && y < 1, prod = prod * b / c];
   }]]];
prod = Simplify[prod];
Print[prod]
```

$\frac{1}{100}$

Circular primes

Problem 35

The number, 197, is called a circular prime because all rotations of the digits: 197, 971, and 719, are themselves prime.

There are thirteen such primes below 100: 2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, and 97.

How many circular primes are there below one million?

```
numcircle = 0;
numprimes = PrimePi[1 000 000];
For[n = 1, n ≤ numprimes, n++,
  {
   temp = IntegerDigits[Prime[n]];
   length = Length[temp];
   For[i = 1, i ≤ length, i++, temp = AppendTo[temp, temp[[i]]]];
   primecheck = 1;
   For[i = 1, i ≤ length, i++,
    {
     num = Sum[10 ^ (length - j) temp[[i - 1 + j]], {j, 1, length}];
     If[PrimeQ[num] == False, {primecheck = 0; i = length + 100;}];
    }];
   If[primecheck == 1, numcircle = numcircle + 1];
  }];
Print[numcircle];
```

55

Goldbach's other conjecture

Problem 46

It was proposed by Christian Goldbach that every odd composite number can be written as the sum of a prime and twice a square.

$9 = 7 + 2\times1^2$
$15 = 7 + 2\times2^2$
$21 = 3 + 2\times3^2$
$25 = 7 + 2\times3^2$
$27 = 19 + 2\times2^2$
$33 = 31 + 2\times1^2$

It turns out that the conjecture was false.

What is the smallest odd composite that cannot be written as the sum of a prime and twice a square?

```
goldbachtest[n_] := Module[{},
  check = 1;
  For[m = 1, m ≤ Sqrt[n/2], m++,
   If[PrimeQ[n - 2 m^2] == True,
     {
      m = 2 n + 10;
      check = 0;
     }];
  ];
  Return[check];
  (*Print[check];*)
 ]
max =   20 000 001;
start = 11;
For[n = start, n ≤ max, n = n + 2,
  {
   (*Print[n, " ",  goldbachtest[n]];*)
   If[PrimeQ[n] == False,
     If[goldbachtest[n] == 1,
       {
        Print["n is ", n];
        n = max + 5;
       }];
    ];
  }];
```

```
n is 5777
```

Self powers

Problem 48

The series, $1^1 + 2^2 + 3^3 + \ldots + 10^{10} = 10405071317$.

Find the last ten digits of the series, $1^1 + 2^2 + 3^3 + \ldots + 1000^{1000}$.

```
Sum[Mod[n^n, 10^10], {n, 1, 1000}]
(* better to do fast exponentiation but... *)
```

```
4 629 110 846 700
```

Combinatoric selections

Problem 53

There are exactly ten ways of selecting three from five, 12345:

123, 124, 125, 134, 135, 145, 234, 235, 245, and 345

In combinatorics, we use the notation, ${}^5C_3 = 10$.

In general,

${}^nC_r = \frac{n!}{r!(n-r)!}$,where $r \le n$, $n! = n\times(n-1)\times\ldots\times3\times2\times1$, and $0! = 1$.

It is not until $n = 23$, that a value exceeds one-million: ${}^{23}C_{10} = 1144066$.

How many, not necessarily distinct, values of ${}^nC_r$, for $1 \le n \le 100$, are greater than one-million?

```
A = 0
Timing[For[n = 1, n ≤ 100, n++, {For[r = 1, r ≤ n / 2, r++,
      {If[Binomial[n, r] > 1 000 000, {If[r == n / 2, A = A + 1, A = A + 2]};]};]};]]
Print["", A, ""]
```

```
0

{0.046800, Null}

4075
```

Powerful digit sum

Problem 56

A googol (10^{100}) is a massive number: one followed by one-hundred zeros; 100^{100} is almost unimaginably large: one followed by two-hundred zeros. Despite their size, the sum of the digits in each number is only 1.

Considering natural numbers of the form, a^b, where $a, b < 100$, what is the maximum digital sum?

```
max = 0;
For[a = 2, a ≤ 100, a++,
  For[b = 2, b ≤ 100, b++,
   {
    temp = IntegerDigits[a^b];
    sum = Sum[temp[[i]], {i, 1, Length[temp]}];
    If[sum > max, max = sum];
   }]];
Print[max]
```

972

Square root convergents

Problem 57

It is possible to show that the square root of two can be expressed as an infinite continued fraction.

$$\sqrt{2} = 1 + 1/(2 + 1/(2 + 1/(2 + \ldots))) = 1.414213\ldots$$

By expanding this for the first four iterations, we get:

$1 + 1/2 = 3/2 = 1.5$
$1 + 1/(2 + 1/2) = 7/5 = 1.4$
$1 + 1/(2 + 1/(2 + 1/2)) = 17/12 = 1.41666\ldots$
$1 + 1/(2 + 1/(2 + 1/(2 + 1/2))) = 41/29 = 1.41379\ldots$

The next three expansions are 99/70, 239/169, and 577/408, but the eighth expansion, 1393/985, is the first example where the number of digits in the numerator exceeds the number of digits in the denominator.

In the first one-thousand expansions, how many fractions contain a numerator with more digits than denominator?

```
count = 0;
For[n = 1, n ≤ 1000, n++,
  {
   x = FromContinuedFraction[ContinuedFraction[Sqrt[2], n]];
   num = Numerator[x];
   denom = Denominator[x];
   If[Floor[Log[10., num]] > Floor[Log[10., denom]], count = count + 1];
  }];
Print[count]
```

153

Totient maximum

Problem 69

Euler's Totient function, φ(n) [sometimes called the phi function], is used to determine the number of numbers less than n which are relatively prime to n. For example, as 1, 2, 4, 5, 7, and 8, are all less than nine and relatively prime to nine, φ(9)=6.

n	Relatively Prime	φ(n)	n/φ(n)
2	1	1	2
3	1,2	2	1.5
4	1,3	2	2
5	1,2,3,4	4	1.25
6	1,5	2	3
7	1,2,3,4,5,6	6	1.1666...
8	1,3,5,7	4	2
9	1,2,4,5,7,8	6	1.5
10	1,3,7,9	4	2.5

It can be seen that n=6 produces a maximum n/φ(n) for $n \leq 10$.

Find the value of $n \leq$ 1,000,000 for which n/φ(n) is a maximum.

```
max = 0;
For[n = 1, n ≤ 1 000 000, n++,
  {
   temp = n / EulerPhi[n];
   If[temp > max,
    {
     max = temp;
     nmax = n;
    }]
  }];
Print[nmax, " ", max];
```

510 510 $\frac{17017}{3072}$