Benford's Law and Number Theory Projects

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https://web.williams.edu/Mathematics/sjmiller/public_html/

https://web.williams.edu/Mathematics/sjmiller/public_html/polymathjrreu/

- **Benford's Law: Video here: https://youtu.be/vs7nQMAF_e8**
    - Benford's law of digit bias states that in many data sets each digit 1 thru 9 is not equally likely to be the leading digit, but rather we observe 1 almost 30% of the time, with the probability falling down to about 4.6% for starting with a 9. In addition to being of theoretical interest, this is also used in detecting various types of fraud. We will explore both aspects. On the fraud side, a lot of our work will be related to new tests to use Benford's law to examine whether or not there has been fraud in election data (I have written a short note with a colleague arguing that the claims circulating on the internet that Benford's law showed there was fraud supporting Biden are wrong; see https://web.williams.edu/Mathematics/sjmiller/public_html/kossovskymiller.pdf). On the theory side, we will explore many topics, including connections with fractal sets (see for example the video here: https://www.youtube.com/watch?v=TMILk79N_Bs). This will be joint with Dan Stoll of Michigan.
    - Go to https://web.williams.edu/Mathematics/sjmiller/public_html/benfordresources/ for some resources of mine on Benford's law.

- **Number Theory: Video here: https://youtu.be/Rn64zHXqDVA (slides here)**
  - Number theory is a vast area with a range of problems. We will focus on projects with minimal pre-requisites that can be split into many sub-problems, so the problems will be accessible to hopefully everyone. Possibilities include further work or generalizations of teh following.
    - Zeckendorf games: This is a game I invented using Fibonacci numbers; a former student proved that Player 2 has a winning strategy but it is a non-constructive proof. Last year one of my Polymath Jr groups extended this to several people and related sequences. See the following papers:
      - The Zeckendorf Game (with Paul Baird-Smith, Alyssa Epstein and Kristen Flint), to appear in the Proceedings of CANT 2018. pdf
      - The Generalized Zeckendorf Game (with Paul Baird-Smith, Alyssa Epstein and Kristen Flint), Fibonacci Quarterly (**57** (2019) no. 5, 1-14) pdf
      - The Fibonacci Quilt Game (with Alexandra Newlon), Fibonacci Quarterly **(2** (2020), 157-168**)** pdf
      - Extending Zeckendorf's Theorem to a Non-constant Recurrence and the Zeckendorf Game on this Non-constant Recurrence Relation (with Ela Boldyriew, Anna Cusenza, Linglong Dai, Pei Ding, Aidan Dunkelberg, John Haviland, Kate Huffman, Dianhui Ke, Daniel Kleber, Jason Kuretski, John Lentfer, Tianhao Luo, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, Yunhao Zhang, Xiaoyan Zheng, and Weiduo Zhu), Fibonacci Quarterly. **(5** (2020), 55-76**)** pdf
      - Deterministic Zeckendorf Games (with Ruoci Li, Xiaonan Li, Clay Mizgerd, Chenyang Sun, Dong Xia, And Zhyi Zhou), Fibonacci Quarterly. **(58** (2020), no. 5, 152-160**)** pdf
      - Winning Strategy for the Multiplayer and Multialliance Zeckendorf Games (with Anna Cusenza, Aiden Dunkelberg, Kate Huffman, Dianhui Ke, Daniel Kleber, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, Xiaoyan Zheng), to appear in the Fibonacci Quarterly. pdf
      - Generalizing Zeckendorf's Theorem to a Non-constant Recurrence (with E. Boldyriew, A. Cusenza, L. Dai, P. Ding, A. Dunkelberg, J. Haviland, K. Huffman, D. Ke, D. Kleber, J. Kuretski, J. Lentfer, T. Luo, C. Mizgerd, V. Tiwari, J. Ye, Y. Zhang, X. Zheng and Weiduo Zhu), to appear in the Fibonacci Quarterly. pdf
    - Non-commutative avoidance: the following preprint is an excellent start, but needs some work to finish and hopefully extend:
      - Avoiding 3-Term Geometric Progressions in Non-Commutative Settings (with Megumi Asada, Eva Fourakis, Eli Goldstein, Sarah Manski, Nathan McNew and Gwyneth Moreland), submitted to the Journal of Integer Sequences. pdf
    - Generalizing the Mordell-Weil group of an elliptic curve to other varieties (this will also involve mentoring some high school students who are working with me). An elliptic curve is of the form $y^2 = x^3 + ax + b$ with a, b integers (and some other conditions to avoid degeneracies). It turns out that one can define a group law on pairs of rationals (x, y) which satisfy this; the goal of this project is to ask related questions in other settings.
    - Math outreach:
      - Math riddle page: I maintain a math riddles page, and could use help in updating it and making it a great resource for teachers and students: http://mathriddles.williams.edu/
      - Math outreach: I have given several lectures to my children, continuing education classes for teachers, ..., and would love to convert these to a book. You can see many of the lectures here: https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html (the first few are ones to my children; initially these were way too long, but it would be great to redo with better slides / graphics).
      - Book project: I am working on a book involving applications of L-functions with a colleague, and if people are interested there is a lot of work that can be done there.
    - Other: Depending on the interest of students and the time that I have, I may add some other problems to the list.

Two L-functions projects, and Artin's primitive root conjecture

Project 1. The objective of this project is to compute the 2-*level density* of zeroes of a certain family of Hecke L-functions. The Katz-Sarnak density conjecture predicts that the main contribution to the asymptotic behavior of such zeroes should be modeled by random matrix theory; we will be interested in finer lower order terms. The *L-functions ratios conjecture* is a conjectural recipe for computing certain averages of products of L-functions in families. We will apply a result predicted by the ratios conjecture to obtain a prediction for lower order terms for the 2-level density of the zeroes of a family of L-functions. A related computation in a different setting may be found at `https://arxiv.org/abs/2001.03265`.

Project 2. The objective of this project is to *numerically* verify predictions for the 1-level density and 2-level density of zeroes of families of L-functions. Our aim would be to produce graphs, like in Figure 6 of `https://www.ams.org/journals/bull/1999-36-01/S0273-0979-99-00766-1/S0273-0979-99-00766-1.pdf`, which numerically verify predictions like found in `https://arxiv.org/pdf/1710.06834.pdf`.

Project 3. For a fixed integer g, when is g a primitive root modulo p? For $g \neq -1$ non-square, Artin's conjecture on primitive roots gives a prediction for the asymptotic density of such primes p. There is no value of g for which the conjecture is known. We will study some variations on this problem. For example: for a pair (g_1, g_2), what is the density of twin primes (p_1, p_2) for which g_i is a primitive root modulo p_i? Recent work has produced a conjectural prediction for this question, based on a probabilistic model for the occurence of such primes. We will build on this and study the k-tuple version of this question.

Complex Analysis:

https://web.williams.edu/Mathematics/sjmiller/public_html/372Fa15/index.htm

- **General Reading for L-functions and Random Matrix Theory**
  - Hayes: The Spectrum of Riemannium: a light description of the connection between random matrix theory and number (there are a few minor errors in the presentation, basically to simplify the story). This is a quick read, and gives some of history.
  - Conrey: L-functions and Random Matrix Theory: This is a high level description of the similarities between number the and random matrix theory.
  - Katz-Sarnak: Zeros of Zeta Functions and Symmetry: Another high level article similar to the others.
  - Diaconis: Patterns in Eigenvalues: this is a bit more readable than the others, and is based on a distinguished lecture he delivered.
  - Miller and Takloo-Bighash: An Invitation to Modern Number Theory: This is the textbook I and a colleague wrote, base years of supervising undergraduate research classes. I know several of you already have a copy -- it will be a good resou the summer, as a lot of the background material we need is readily available here. Particularly important chapters for us
    - chapter15 (which discusses the connections between random matrix theory and number theory, and is available onli
    - chapter 18 (which does the explicit formula for Dirichlet characters);
    - chapter 3 (which reviews L-functions and is also online).
  - Firk and Miller: Nuclei, primes and the Random Matrix connection: a survey paper on the history of the subject, includi both the nuclear physics experiments and the theoretical calculations.