Miller: Polymath Jr Number Theory Project

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Polymath Jr Page:

https://web.williams.edu/Mathematics/sjmiller/public_html/polymathjrreu/

Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/

Number Theory

- Number theory is a vast area with a range of problems. We will focus on projects with minimal pre-requisites that can be split into many sub-problems, so the problems will be accessible to hopefully everyone. Possibilities include further work or generalizations of teh following.
  - Zeckendorf games: This is a game I invented using Fibonacci numbers; a former student proved that Player 2 has a winning strategy but it is a non-constructive proof. Last year one of my Polymath Jr groups extended this to several people and related sequences. See the following papers:
    - The Zeckendorf Game (with Paul Baird-Smith, Alyssa Epstein and Kristen Flint), to appear in the Proceedings of CANT 2018. pdf
    - The Generalized Zeckendorf Game (with Paul Baird-Smith, Alyssa Epstein and Kristen Flint), Fibonacci Quarterly (**57** (2019) no. 5, 1-14) pdf
    - The Fibonacci Quilt Game (with Alexandra Newlon), Fibonacci Quarterly **(2** (2020), 157-168**)** pdf
    - Extending Zeckendorf's Theorem to a Non-constant Recurrence and the Zeckendorf Game on this Non-constant Recurrence Relation (with Ela Boldyriew, Anna Cusenza, Linglong Dai, Pei Ding, Aidan Dunkelberg, John Haviland, Kate Huffman, Dianhui Ke, Daniel Kleber, Jason Kuretski, John Lentfer, Tianhao Luo, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, Yunhao Zhang, Xiaoyan Zheng, and Weiduo Zhu), Fibonacci Quarterly. **(5** (2020), 55-76**)** pdf
    - Deterministic Zeckendorf Games (with Ruoci Li, Xiaonan Li, Clay Mizgerd, Chenyang Sun, Dong Xia, And Zhyi Zhou), Fibonacci Quarterly. **(58** (2020), no. 5, 152-160**)** pdf
    - Winning Strategy for the Multiplayer and Multialliance Zeckendorf Games (with Anna Cusenza, Aiden Dunkelberg, Kate Huffman, Dianhui Ke, Daniel Kleber, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, Xiaoyan Zheng), to appear in the Fibonacci Quarterly. pdf
    - Generalizing Zeckendorf's Theorem to a Non-constant Recurrence (with E. Boldyriew, A. Cusenza, L. Dai, P. Ding, A. Dunkelberg, J. Haviland, K. Huffman, D. Ke, D. Kleber, J. Kuretski, J. Lentfer, T. Luo, C. Mizgerd, V. Tiwari, J. Ye, Y. Zhang, X. Zheng and Weiduo Zhu), to appear in the Fibonacci Quarterly. pdf
  - Non-commutative avoidance: the following preprint is an excellent start, but needs some work to finish and hopefully extend:
    - Avoiding 3-Term Geometric Progressions in Non-Commutative Settings (with Megumi Asada, Eva Fourakis, Eli Goldstein, Sarah Manski, Nathan McNew and Gwyneth Moreland), submitted to the Journal of Integer Sequences. pdf
  - Generalizing the Mordell-Weil group of an elliptic curve to other varieties (this will also involve mentoring some high school students who are working with me). An elliptic curve is of the form $y^2 = x^3 + ax + b$ with a, b integers (and some other conditions to avoid degeneracies). It turns out that one can define a group law on pairs of rationals (x, y) which satisfy this; the goal of this project is to ask related questions in other settings.
  - Math outreach:
    - Math riddle page: I maintain a math riddles page, and could use help in updating it and making it a great resource for teachers and students: http://mathriddles.williams.edu/
    - Math outreach: I have given several lectures to my children, continuing education classes for teachers, ..., and would love to convert these to a book. You can see many of the lectures here: https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html (the first few are ones to my children; initially these were way too long, but it would be great to redo with better slides / graphics).
    - Book project: I am working on a book involving applications of L-functions with a colleague, and if people are interested there is a lot of work that can be done there.
  - Other: Depending on the interest of students and the time that I have, I may add some other problems to the list.

Zeckendorf games (video here: https://www.youtube.com/watch?v=S-t_455zfb8): This is a game I invented using Fibonacci numbers; a former student proved that Player 2 has a winning strategy but it is a non-constructive proof. Last year one of my Polymath Jr groups extended this to several people and related sequences. See the following papers:

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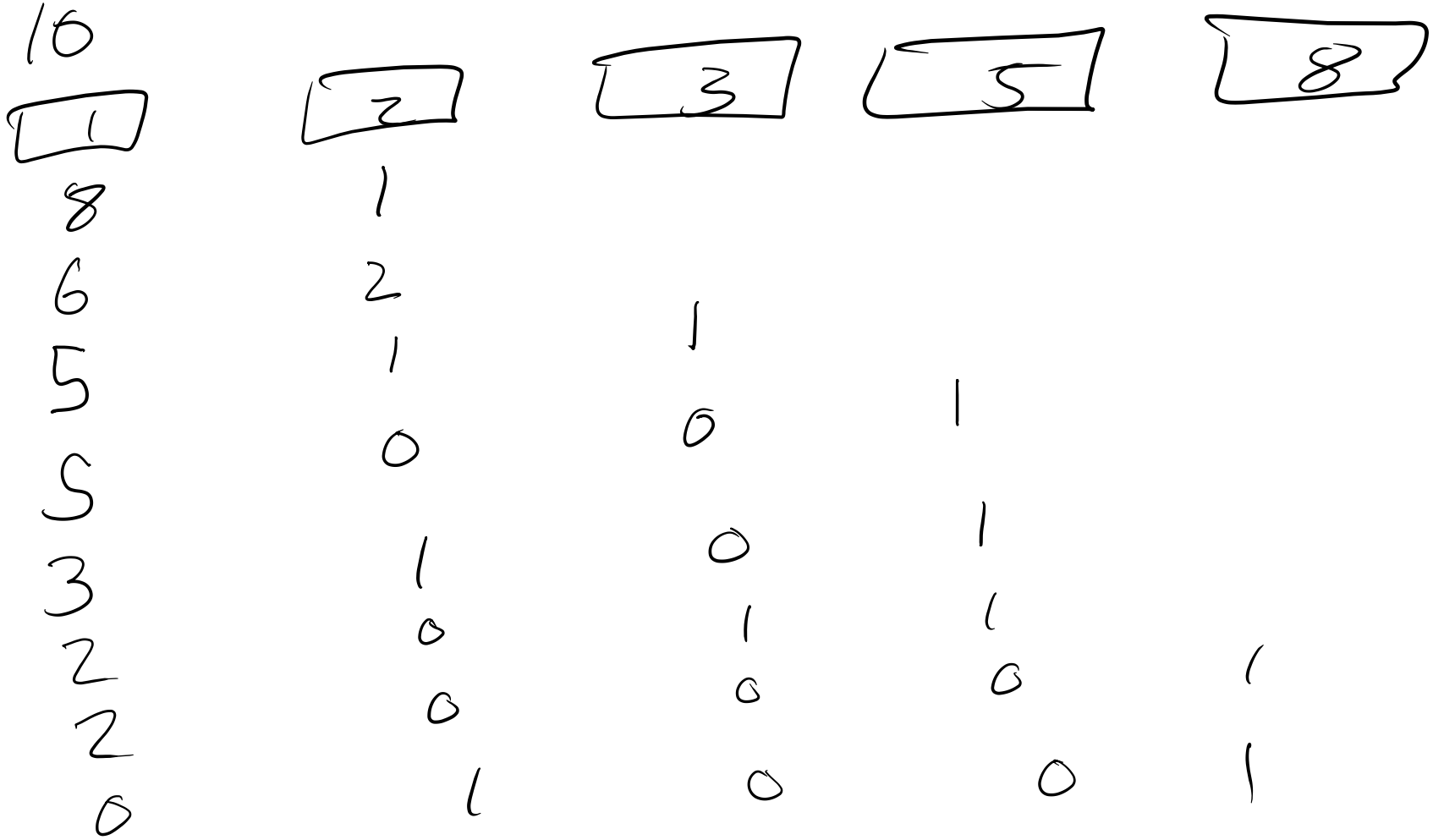
1, 2, 3, 5, 8, 13, ... $F_{n+1} = F_n + F_{n-1}$

$2F_n = F_{n+1} + F_{n-2}$

$2F_1 = 1 \cdot F_2$

$F_n + F_{n-1} = F_{n+1}$



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$1, r, r^2 \qquad a, ar, ar^2 \qquad 4, 6, 9$

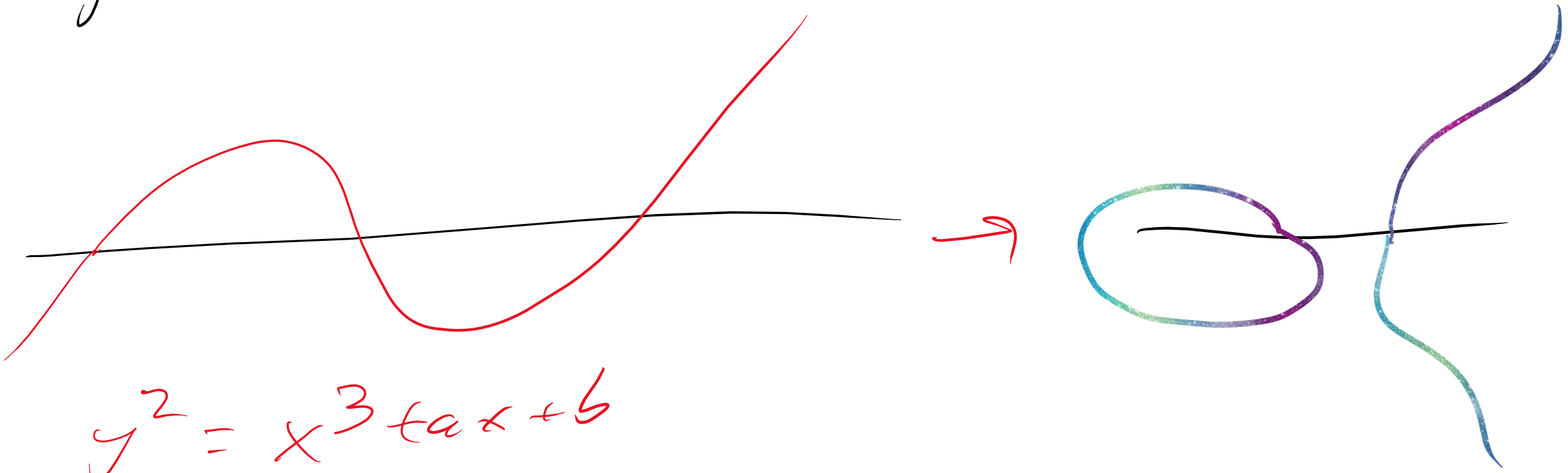
$a_1, a_2, a_3, \ldots \qquad \lim_{n \to \infty} \frac{a_n}{n}$

$1, 2, 3, 5, 6, 7, 8, \ldots$ Greedy

Generalizing the Mordell-Weil group of an elliptic curve to other varieties (this will also involve mentoring some high school students who are working with me). An elliptic curve is of the form $y^2 = x^3 + ax + b$ with a, b integers (and some other conditions to avoid degeneracies). It turns out that one can define a group law on pairs of rationals (x, y) which satisfy this; the goal of this project is to ask related questions in other settings.

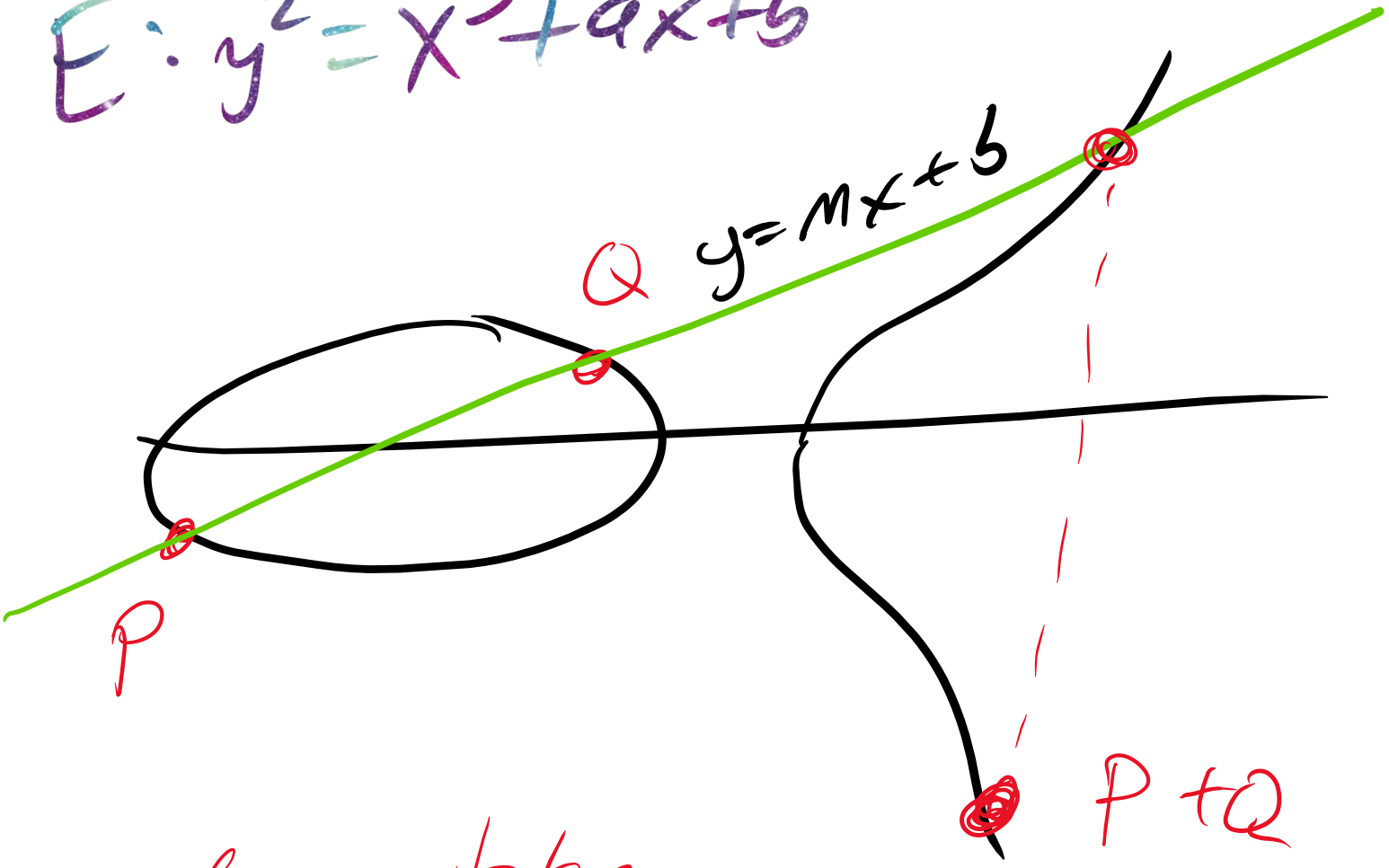
$y^2 = x^3 + ax + b \quad a, b \in \mathbb{Z}$

$y = x^3 + ax + b$



$y^2 = x^3 + ax + b$

$E: y^2 = x^3 + ax + b$



Commutative Group

$P + Q = Q + P$

non-trivial to prove associativity

$E(Q) = \{(x, y) : x, y$ rational and $y^2 = x^3 + ax + b\}$

P, Q rational $\Rightarrow P + Q$

$\Rightarrow m, b$ rational

$(mx + b)^2 = x^3 + ax + b$

$x^3 - (mx + b)^2 + ax + b = (x - x_P)(x - x_Q)(x - x_{P+Q})$

L-Functions and Random Matrix Theory:

•Introductory lectures on Random Matrix Theory and L-functions:

- Part I (Classical RMT, Intro L-fns, Dirichlet): http://youtu.be/2PuUbk6gUMM (slides: part 1)
- Part II (Convolving families, cusp forms: slides here): http://youtu.be/vJz6W24tDik (slides part 2)

•From Sato-Tate Distributions to Low-Lying Zeros: http://youtu.be/VBzVAvZ6k6A (slides here).

•From the Manhattan Project to Elliptic Curves: MASON IV (3/7/20). pdf (video here: https://youtu.be/p15X3ERNvLs)

•Introduction to L-functions for SMALL students:
2021: https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/intronumbertheory/

- two level density for Hecke L-functions (to use the Ratios Conjecture work from some of my SMALL REU students)
- Numerical evidence for one-level density lower order term calculations
- Artin Hardy Littlewood Conjecture (a nice exercise in elementary analytic number theory)