NUMBERS WITH SMALL/LARGE DIVISORS SATISFYING SOME RECURRENCE RELATION

ABSTRACT. I suggest some possible projects, where we study numbers whose divisors satisfy certain recurrence relation. I also give some contexts for open-ended projects regarding Schreier sets and what I call Schreier numbers.

1. PROJECT 1: ELEMENTARY NUMBER THEORY FLAVOR

Let N be a positive integers. The set of small divisors of N is

$$S_N := \{n \leq \sqrt{N} : n|N\}$$

and the set of large divisors of N is

$$L_N := \{n \geq \sqrt{N} : n|N\}.$$

If we do not want to include the obvious divisors, namely 1 and N, we can define the corresponding set of "nontrivial" small/large divisors

$$S'_N := \{n \leq \sqrt{N} : n > 1, n|N\} \text{ and } L'_N := \{n \geq \sqrt{N} : n|N, n < N\}.$$

In 2018, Iannucci [11] characterized all positive integers N with S_N in an arithmetic progression (see [11, Theorem 4].) Continuing the work, Chu [4] characterized all positive integers N with L'_N in arithmetic progression (see [4, Theorem 1].) Recently, Chentouf [3] generalized Iannucci's result by considering a more general recurrence relation than arithmetic progression, namely *linear recurrence of order at most* 2 (see [3, (2)] for the definition) and characterized all integers N with S_N satisfying the recurrence.

Project goal #1: Characterize all integers N with L'_N satisfying a linear recurrence of order at most 2, which generalizes [4, Theorem 1] **the same way Chentouf generalizd** [11, Theorem 4]**. See also the discussion at the beginning of page 5 of** [3]**.**

Let us now describe Project goal #2. In characterizing all numbers N whose S_N are in arithmetic progression, the trivial divisor 1 played a crucial role in Iannucci's argument. Motivated by this, [5, Theorem 1.1] characterized all numbers N whose S'_N are in arithmetic progression. The argument for [5, Theorem 1.1] is a bit more involved. In the same manner, the trivial divisor 1 was used in the argument of Chentouf. So, the next goal is

Project goal #2: Characterize all integers N with S'_N satisfying a linear recurrence of order at most 2.

Project goal #3: What are some interesting (not too rigid) structure to put on S'_N and L'_N? Can we characterize these N? (This is an open-ended project.)

2. PROJECT 2: COMBINATORIAL FLAVOR

2.1. **Background.** A set $A \subset \mathbb{N}$ is said to be *Schreier* if either A is empty or $\min A \geq |A|$, where $|A|$ is the cardinality of set A. For example, $\{2, 3\}$ and $\{4, 7, 20\}$ are Schreier sets, while $\{1, 2\}$ and $\{4, 5, 10, 20, 30\}$ are not. Schreier sets have their roots in Banach space theory and Ramsey-type theorems for subsets of $\mathbb{N}$.

The Fibonacci sequence is defined as follows: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. A. Bird [2] showed that for each $n \geq 1$, if we let

$$\mathcal{A}_n \ := \ \{A \subset \{1, \ldots, n\} \ : \ n \in A \text{ and } \min A \geq |A|\},$$

then $|\mathcal{A}_n| = F_n$. This was quite a neat and surprising result. See [2] for a bijective proof. Following the discovery by A. Bird, there has been research on various recurrences produced by counting Schreier-type sets (see [1, 7, 8, 9, 10, 6, 12]). In particular, [6] showed another way to obtain the Fibonacci sequence from counting Schreier sets. [1] studied the more general condition $p \min A \geq q|A|$, where $p, q \geq 1$ and showed a relation between Turán graphs and Schreier-type sets. [7] gave a combinatorial proof of the above relation and generalized the relation by modifying Tuán graphs. Finally, [8, 9, 10, 12] showed other recurrences from couting Schreier-type sets.

2.2. **Project goals.** An open question is whether we can produce the tribonacci numbers (see A000073) from couting Schreier sets in certain way. What about tetranacci (see A000078) numbers? Besides, we can explore what other interesting sequences that counting Schreier-type sets give us.

3. PROJECT 3: ANALYTIC/ELEMENTARY NUMBER THEORY FLAVOR

3.1. **Definitions.**

Definition 3.1. A positive integers N is said to be Schreier if S'_N is Schreier. A number is said to be non-Schreier if it is not Schreier.

Definition 3.2. Let S be a set of positive integers. The upper and lower, respectively, density of S is

$$\overline{d}(S) \ = \ \limsup_{N\to\infty} \frac{|\{n \in S : n \leq N\}|}{N}$$

and

$$\underline{d}(S) \ = \ \liminf_{N\to\infty} \frac{|\{n \in S : n \leq N\}|}{N}.$$

If $\overline{d}(S) = \underline{d}(S) = d$, we say that S has density d and write $d(S) = d$.

Remark 3.3. For all N, the set L_N is Schreier. Indeed, $\min L_N \ \geq \ \sqrt{N}$, while

$$|L_N| \ = \ |S_N| \ \leq \ \sqrt{N}.$$

Hence, it only makes sense to define a Schreier number by requiring the set of small divisors to be Schreier.

3.2. **Observations and questions.**

Example 3.4. *All primes are Schreier. The product of two primes is also Schreier.*

Proof. For each prime p, $S_p = \emptyset$ and so, p is Schreier. If $n = pq$, where $p < q$ are primes, then $S_n = \{p\}$, which is Schreier. □

Proposition 3.5. *The gap between two consecutive non-Schreier numbers is at most* 6. *Furthermore, for* $k = 1, 2$*, a gap of* k *appears infinitely often.*

Proof. Let N be a non-Schreier number. We can assume that $N \geq 36$. (The claim holds for $N \leq 36$.) If $N \equiv k \bmod 6$, for some $0 \leq k \leq 5$, then $6|(N + 6 - k)$ and so, $\{2, 3, 6\} \subset S_{N+6-k}$. Hence, $N + 6 - k$ is non-Schreier. Therefore, the gap between N and the next non-Schreier number is at most $6 - k$.

We now show that a gap of 1 appears infinitely often. Consider the pair of integers $(405+630k, 406+630k)$ for some $k \geq 1$. Since $\{3, 5, 9, 15\} \subset S_{405+630k}$ for $k \geq 1$, we know that $405 + 630k$ is non-Schreier for $k \geq 1$. On the other hand, since $\{2, 7, 14\} \subset S_{406+630k}$ for $k \geq 1$, $406 + 630k$ is non-Schreier for $k \geq 1$. Therefore, for each $k \geq 1$, $(405 + 630k, 406 + 630k)$ is a pair of non-Schreier numbers that are 1 apart.

Next, we show that a gap of 2 appears infinitely often. Consider the pair of integers $(40 + 70k, 42 + 70k)$ for some $k \geq 3$. Since $\{2, 5, 10\} \subset S_{40+70k}$ and $\{2, 7, 14\} \subset S_{42+70k}$, both integers are non-Schreier. By the Dirichlet's theorem for primes in arithmetic progression, we know that $41 + 70k$ is prime and thus, is Schreier for infinitely many k. We conclude that $40 + 70k$ and $42 + 70k$ are consecutive non-Schreier integers infinitely often. □

Corollary 3.6. *Let* S *be the set of all non-Schreier numbers. Then* $\underline{d}(S) \geq 1/6$.

Problem 3.7. Can we improve the lower bound of $1/6$? Can we show $d(S)$ actually exist?

Problem 3.8. Let $\overline{S}$ be the set of all Schreier numbers. What can we say about $\underline{d}(\overline{S})$?

Problem 3.9. Show that for any $3 \leq k \leq 6$, a gap of k between consecutive non-Schreier numbers appears infinitely often.

Problem 3.10. Show that there are arbitrarily large gaps between two Schreier numbers.

Proposition 3.11. *There are arbitrarily long arithmetic progressions of Schreier numbers. There are arbitrary long arithmetic progressions of non-Schreier numbers.*

Proof. From the Green-Tao theorem, we know that there are arbitrarily long arithmetic progressions of primes in arithmetic progression. Since primes are Schreier, we have arbitrarily long arithmetic progressions of Schreier numbers.

For arbitrarily long arithmetic progressions of non-Schreier numbers, consider the sequence $(36 + 6n)_{n=1}^{\infty}$. For each $n \geq 1$, we have $2, 3, 6 \in S_{36+6n}$. Hence, $36 + 6n$ is non-Schreier. □

Problem 3.12. Can we show that there are arbitrarily long arithmetic progressions of non-Schreier numbers without using the Green-Tao theorem?

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