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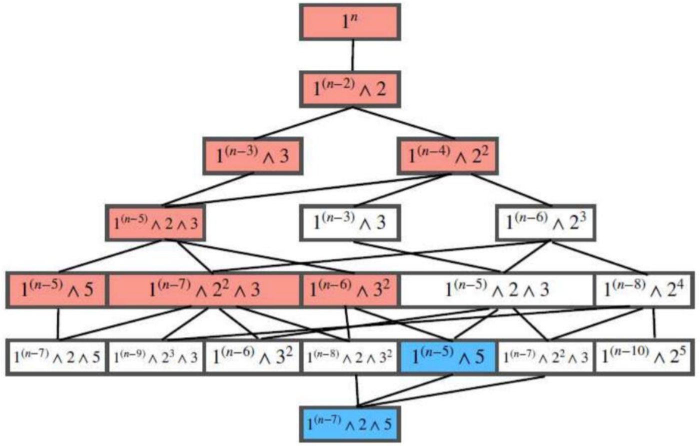
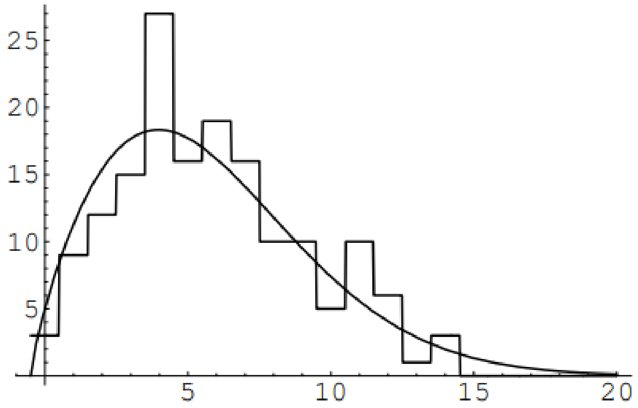
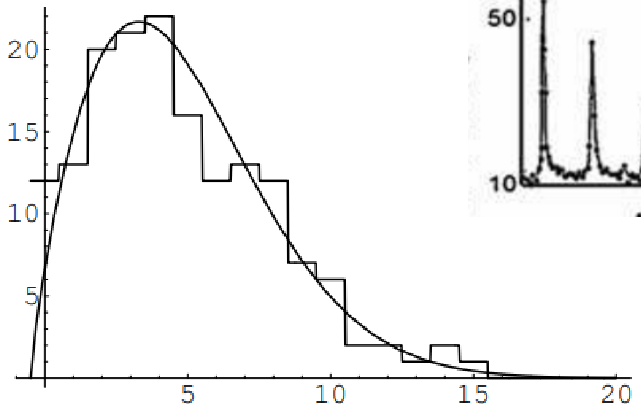


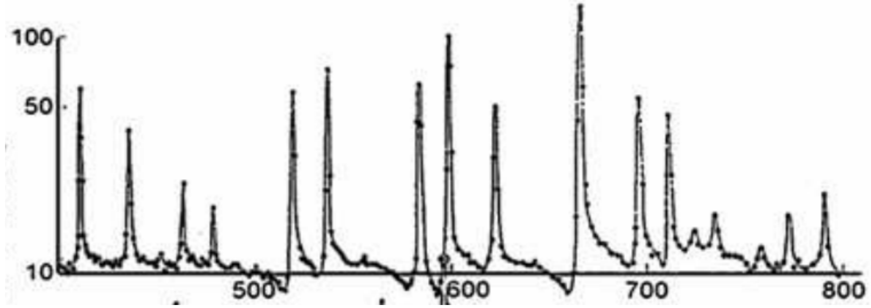
Polymath Number Theory: Summer 2022

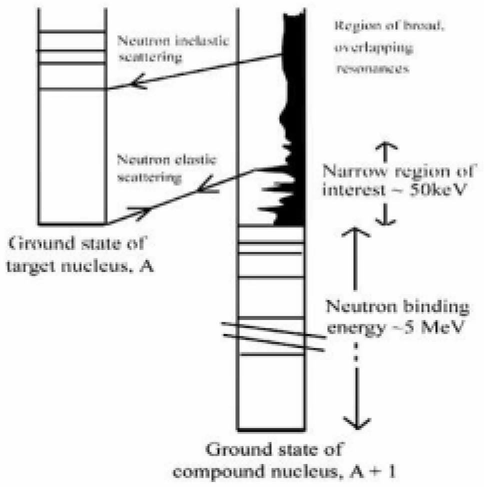
https://web.williams.edu/Mathematics/sjmiller/public_html/polymathreumiller/

Talk: https://youtu.be/yXuKcI0BfbQ









Polymath-REU 2022: Miller Proposed Projects

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Miller's homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/

Greetings. I am excited to be involved in the inaugural Polymath-REU Program. I currently serve as the Director of the Williams SMALL REU, and have been mentoring students for over two decades. I earned a PhD from Princeton working with Peter Sarnak and Henryk Iwaniec in analytic number theory (specifically, low-lying zeros for families of elliptic curves). Below are some general areas of problems I am proposing for the Polymath-REU project. Different projects will be supervised with different colleagues of mine.

Ramsey Theory: Years ago some of my SMALL REU students and I looked at non-commutative versions of some standard problems in the field, specifically avoiding 3-term geometric progressions. We have working notes here. Most of a paper is done, and in addition to finishing things off there are opportunities to explore related problems.

Classically, there has been interest in how large a set can be while still avoiding arithmetic or geometric progressions. In a 1961 paper Rankin [Ran] introduced the idea of considering how large a set of integers can be without containing terms which are in geometric progression. He constructed a subset of the integers which avoids 3-term geometric progressions and has asymptotic density approximately 0.719745. Brown and Gordon [BG] noted that the set Rankin considered was the set obtained by greedily including integers subject to the condition that such integers do not create a progression involving integers already included in the set.

This question has been generalized to number fields [BHMMPTW] and polynomial rings over finite fields [AFGMMMM]. The purpose of [BHMMPTW] was to see how changing from subsets of $\mathbb{Z}$ to subsets of number fields affected the answer, while in [AFGMMMM] it was to see how the extra combinatorial structure of $\mathbb{F}_q[x]$ affected the tractability and features of the problem. In our case, we wish to see how non-commutativity affects the answer.

The first half of this paper (Sections 2 through 5) is dedicated to studying the problem in the Hurwitz order quaternions, Q_{Hur} (see Section 2 for a review of their properties). We consider sets avoiding geometric progression of the form a, ar, ar^2 with $a, r \in Q_{\text{Hur}}$, being careful to specify the order of multiplication due to the non-commutativity of the algebra. We produce some bounds on the supremum of upper densities of sets avoiding 3-term geometric progressions, and use Rankin's greedy set to construct a similar set avoiding 3-term geometric progressions in the Hurwitz order quaternions. We also discuss the peculiarities of this setting in Section 5. The second half (Section 6) is dedicated to studying the question in the setting of free groups. We arrive at the following results.

Theorem 3.1. *Let m_{Hur} be supremum of upper densities of subsets of Q_{Hur} containing no 3-term geometric progressions. Then*

$$.946589 \leq m_{\text{Hur}} \leq .952381. \tag{1.1}$$

Theorem 4.2. *Let Q_{Ran} be the set of Hurwitz quaternions with norm in Rankin's greedy set (avoiding 3-term geometric progressions in $\mathbb{Z}$). Let $A_3^*(\mathbb{Z})$ be the greedy set avoiding 3-term arithmetic progressions. The asymptotic density of Q_{Ran} is*

$$d(Q_{\text{Ran}}) = \left(\prod_{p \text{ odd}} \left[\sum_{n \in A_3^*(\mathbb{Z})} \frac{p^{n+3} - p^{n+2} - p^2 + 1}{p^2(p-1)p^{2n}}\right]\right) \cdot \left(\sum_{n \in A_3^*(\mathbb{Z})} \frac{2^2 - 1}{2^2 2^{2n}}\right) \approx 0.771245. \tag{1.2}$$

Theorem 6.2. *Let $\mathcal{G} = \langle x, y : x^2 = y^2 = 1 \rangle$ be the free group on two generators each of order two. Order the group as $W = (I, x, y, xy, yx, xyx, yxy, xyxy, yxyx, \dots)$ and take the set G formed by greedily taking elements that don't form a 3-term progression with previously added ones. Then*

$$\frac{|G \cap \{w \in W : \text{length}(w) \leq 2 \cdot 3^n\}|}{|\{w \in W : \text{length}(w) \leq 2 \cdot 3^n|\}|} = \frac{2^{n+1}}{1 + 4 \cdot 3^n}, \tag{1.3}$$

and in general

$$\frac{|G \cap \{w \in W : \text{length}(w) \leq n\}|}{|\{w \in W : \text{length}(w) \leq n\}|} = \Theta\left((2/3)^{\log_3 n}\right). \tag{1.4}$$

Elementary Number Theory: Small/large divisors satisfying recurrence relations:

These projects are from a former SMALL REU student of mine, Hung Viet Chu at Illinois; see here for full details; one of the problems is pasted to the right.

Let N be a positive integers. The set of small divisors of N is

$$S_N := \{n \leq \sqrt{N} : n|N\}$$

and the set of large divisors of N is

$$L_N := \{n \geq \sqrt{N} : n|N\}.$$

If we do not want to include the obvious divisors, namely 1 and N, we can define the corresponding set of “nontrivial" small/large divisors

$$S'_N := \{n \leq \sqrt{N} : n > 1, n|N\} \text{ and } L'_N := \{n \geq \sqrt{N} : n|N, n < N\}.$$

In 2018, Iannucci [11] characterized all positive integers N with S_N in an arithmetic progression (see [11, Theorem 4].) Continuing the work, Chu [4] characterized all positive integers N with L'_N in arithmetic progression (see [4, Theorem 1].) Recently, Chentouf [3] generalized Iannucci's result by considering a more general recurrence relation than arithmetic progression, namely *linear recurrence of order at most* 2 (see [3, (2)] for the definition) and characterized all integers N with S_N satisfying the recurrence.

Project goal #1: Characterize all integers N with L'_N satisfying a linear recurrence of order at most 2, which generalizes [4, Theorem 1] the same way Chentouf generalizd [11, Theorem 4]. See also the discussion at the beginning of page 5 of [3].

Let us now describe Project goal #2. In characterizing all numbers N whose S_N are in arithmetic progression, the trivial divisor 1 played a crucial role in Iannucci's argument. Motivated by this, [5, Theorem 1.1] characterized all numbers N whose S'_N are in arithmetic progression. The argument for [5, Theorem 1.1] is a bit more involved. In the same manner, the trivial divisor 1 was used in the argument of Chentouf. So, the next goal is

Project goal #2: Characterize all integers N with S'_N satisfying a linear recurrence of order at most 2.

Project goal #3: What are some interesting (not too rigid) structure to put on S'_N and L'_N? Can we characterize these N? (This is an open-ended project.)

f-palindromes: These projects are from a colleague of mine, Daniel Tsai, at Nagoya University; see here.

The concept of v-palindromes is introduced in [1] and subsequently four manuscripts [3, 2, 5, 4] were written about them. Consider the number 198 whose digit reversal is 891. Their prime factorizations are

$$198 = 2 \cdot 3^2 \cdot 11, \tag{1}$$

$$891 = 3^4 \cdot 11, \tag{2}$$

and we have

$$2 + (3 + 2) + 11 = (3 + 4) + 11. \tag{3}$$

In other words, the sum of the numbers "appearing" on the right-hand-side of (1) equals that of (2). We now define v-palindromes rigorously, but our definition is slightly different from that in [1, 3, 2, 5, 4].

Definition 1. Let $b \geq 2$, $L \geq 1$, and $0 \leq a_0, a_1, \ldots, a_{L-1} < b$ be any integers. We denote

$$(a_{L-1} \cdots a_1 a_0)_b = \sum_{i=0}^{L-1} a_i b^i. \tag{4}$$

Definition 2. Let the base $b \geq 2$ representation of an integer $n \geq 1$ be $(a_{L-1} \cdots a_1 a_0)_b$. The *$b$-reverse* of n is defined to be

$$r_b(n) = (a_0 a_1 \cdots a_{L-1})_b. \tag{5}$$

So for example $r_{10}(198) = 891$.

Definition 3. Let $f \colon \mathbb{N} \to \mathbb{C}$ be any function and $b \geq 2$ an integer. An integer $n \geq 1$ is an *f-palindrome in base b* if $f(n) = f(r_b(n))$. If in addition $n \neq r_b(n)$, then n is a *nonpalindromic f-palindrome in base b*.

Definition 4. The additive function $v \colon \mathbb{N} \to \mathbb{Z}$ is defined by setting $v(p) = p$ for primes p and $v(p^\alpha) = p + \alpha$ for prime powers p^α with $\alpha \geq 2$.

With these definitions, 198 is a nonpalindromic v-palindrome in base 10. We explain the naming "palindrome". If $f = \mathrm{id}_{\mathbb{N}}$ (or is just injective), then an f-palindrome in base b is simply a palindrome in base b.

The following are sequences of nonpalindromic v-palindromes in base 10.

$$18, 198, 1998, \ldots, \tag{6}$$

$$18, 1818, 181818, \ldots. \tag{7}$$

In (6), we simply keep increasing the number of 9's in the middle; in (7), we simply keep concatenating another 18.

Influenced by (6), we propose the following problem.

Problem 1. Try to find other sequences like (6), where we simply keep increasing the number of one of the digits, all of whose terms are nonpalindromic f-palindromes in base b, for the same f and b.

Zeckendorf Games: Baird-Smith, Epstein, Flint and Miller devised a game based on the Fibonacci numbers (1, 2, 3, 5, ... and in general $F_{n+1} = F_n + F_{n-1}$) and one of their interesting properties, the Zeckendorf Decomposition (every integer can be written uniquely as a sum of non-adjacent Fibonacci numbers). It was proved that every game terminates, and a non-constructive proof shows that Player Two always has a winning strategy if the starting value is at least 3). There are still many open questions about this game and its generalizations. See https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ZeckGameCANT10.pdf, https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ZeckGameGeneral_FibQ10.pdf and https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/FQgame30.pdf. I have a 40 minute talk on the subject: From Monovariants to Zeckendorf Decompositions and Games, and Random Matrix Theory, Williams College (7/14/21) and Texas Tech (7/29/21). pdf (video: https://youtu.be/Kayru_V75V8)