f-PALINDROMES

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The concept of v-palindromes is introduced in [1] and subsequently four manuscripts [3, 2, 5, 4] were written about them. Consider the number 198 whose digit reversal is 891. Their prime factorizations are

$$198 = 2 \cdot 3^2 \cdot 11, \tag{1}$$

$$891 = 3^4 \cdot 11, \tag{2}$$

and we have

$$2 + (3 + 2) + 11 = (3 + 4) + 11. \tag{3}$$

In other words, the sum of the numbers "appearing" on the right-hand-side of (1) equals that of (2). We now define v-palindromes rigorously, but our definition is slightly different from that in [1, 3, 2, 5, 4].

Definition 1. Let $b \geq 2$, $L \geq 1$, and $0 \leq a_0, a_1, \ldots, a_{L-1} < b$ be any integers. We denote

$$(a_{L-1} \cdots a_1 a_0)_b = \sum_{i=0}^{L-1} a_i b^i. \tag{4}$$

Definition 2. Let the base $b \geq 2$ representation of an integer $n \geq 1$ be $(a_{L-1} \cdots a_1 a_0)_b$. The *$b$-reverse* of n is defined to be

$$r_b(n) = (a_0 a_1 \cdots a_{L-1})_b. \tag{5}$$

So for example $r_{10}(198) = 891$.

Definition 3. Let $f\colon \mathbb{N} \to \mathbb{C}$ be any function and $b \geq 2$ an integer. An integer $n \geq 1$ is an *f-palindrome in base b* if $f(n) = f(r_b(n))$. If in addition $n \neq r_b(n)$, then n is a *nonpalindromic f-palindrome in base b.*

Definition 4. The additive function $v\colon \mathbb{N} \to \mathbb{Z}$ is defined by setting $v(p) = p$ for primes p and $v(p^\alpha) = p + \alpha$ for prime powers p^α with $\alpha \geq 2$.

With these definitions, 198 is a nonpalindromic v-palindrome in base 10. We explain the naming "palindrome". If $f = \mathrm{id}_{\mathbb{N}}$ (or is just injective), then an f-palindrome in base b is simply a palindrome in base b.

The following are sequences of nonpalindromic v-palindromes in base 10.

$$18, 198, 1998, \ldots, \tag{6}$$

$$18, 1818, 181818, \ldots. \tag{7}$$

In (6), we simply keep increasing the number of 9's in the middle; in (7), we simply keep concatenating another 18.

Influenced by (6), we propose the following problem.

Problem 1. Try to find other sequences like (6), where we simply keep increasing the number of one of the digits, all of whose terms are nonpalindromic f-palindromes in base b, for the same f and b.

The sequence (7) inspired [1, Theorem 1], which was subsequently developed into the more extensive [3]. We make the following definition.

Definition 5. Let the base $b \geq 2$ representation of an integer $n \geq 1$ be $(a_{L-1} \cdots a_1 a_0)_b$. For integers $k \geq 1$, we define

$$n(k)_b = (\underbrace{a_{L-1} \cdots a_1 a_0 \, a_{L-1} \cdots a_1 a_0 \ \cdots\cdots \ a_{L-1} \cdots a_1 a_0}_{k \text{ copies of } a_{L-1} \cdots a_1 a_0})_b. \tag{8}$$

So for example $18(3)_{10} = 181818$. According to [3], statements such as the following theorems can be proved.

Theorem 1. *For integers $k \geq 1$, the number $13(k)_{10}$ is a v-palindrome in base 10 if and only if* (i) $6045 \mid k$ *or* (ii) $15 \mid k$ *but* $13 \nmid k$ *and* $31 \nmid k$.

Theorem 2. *For integers $k \geq 1$, the number $17(k)_{10}$ is a v-palindrome in base 10 if and only if* (i) $33790 \mid k$ *or* (ii) $280 \mid k$ *but* $7 \nmid k$ *and* $71 \nmid k$.

We propose the following problem.

Problem 2. While [3] is only for v-palindromes in base 10, the same reasoning might also work for f-palindromes in base b, for other choices of f and b. Try to characterise those choices of f and b such that the theory of [3] holds.

Finally, we also mention that the sequences (6) and (7) are actually parts of a larger family of nonpalindromic v-palindromes in base 10 indicated in [2, Theorem 3].

References

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