On the polynomial $x(x+1)(x+2)(x+3)$

Warren Sinnott, Steven J Miller, Cosmin Roman

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Abstract

We show that $x(x+1)(x+2)(x+3)$ is never a perfect square or cube for x a positive integer. The first author is responsible for showing it is never a square; the second author did the straightforward generalization to show it is never a cube. This involves using elliptic curves to handle some cases; without using elliptic curves, one can handle many cases by reducing to the Catalan equation. The third author showed how to generalize the Catalan argument to handle the remaining cases, which the second and third authors then generalized to show that $x(x+1)(x+2)(x+3)$ is never a perfect power for any integer x.

1 $x(x+1)(x+2)(x+3)$ is not a square

We consider the question of whether

$$x(x+1)(x+2)(x+3) \ = \ y^2 \tag{1}$$

has any solutions in positive integers. (We find that it does not.) Let

$$u \ = \ 2x+3, \quad z \ = \ u^2 \tag{2}$$

so that

$$\begin{aligned} (4y)^2 &= 2x(2x+2)(2x+4)(2x+6) \\ &= (u-3)(u-1)(u+1)(u+3) \\ &= (u^2-1)(u^2-9) \\ &= (z-1)(z-9). \end{aligned} \tag{3}$$

The difference between $z-1$ and $z-9$ is 8, so the factors $z-1$ and $z-9$ have at most a power of 2 in common; since the left-hand side of the equations above is a square we may write

$$z-1 \;=\; 2^a v^2, \quad z-9 \;=\; 2^b w^2, \tag{4}$$

where a, b are either 0 or 1 and $a+b$ is even, i.e., either $a=b=0$ or $a=b=1$.

Case One: $a=b=0$**.** Here we have

$$z \;=\; 1+v^2 \;=\; 9+w^2, \tag{5}$$

so

$$8 \;=\; v^2-w^2 \;=\; (v-w)(v+w). \tag{6}$$

So $v-w$ and $v+w$ are divisors of 8, the second larger than the first; also $v-w$ and $v+w$ must have the same parity. The only possibility is then

$$v-w \;=\; 2, \quad v+w \;=\; 4, \tag{7}$$

which implies that $v=3$, and $z=10$. But $z=u^2$ is a square, so there are no solutions in this case.

Case Two: $a=b=1$**.** Here we have

$$z \;=\; 1+2v^2 \;=\; 9+2w^2, \tag{8}$$

so

$$4 \;=\; v^2-w^2 \;=\; (v-w)(v+w). \tag{9}$$

So $v-w$ and $v+w$ are divisors of 4, the second larger than the first, and both of the same parity; so there are no solutions in this case either.

2 $x(x+1)(x+2)(x+3)$ is not a cube

We argue similarly as before. We consider the question of whether

$$x(x+1)(x+2)(x+3) \;=\; y^3 \tag{10}$$

has any solutions in positive integers. (We find that it does not.) Let

$$u \;=\; 8x+12, \quad z \;=\; u^2. \tag{11}$$

We find

$$\begin{aligned} 8^4 y^3 &= 8x(8x+8)(8x+16)(8x+24) \\ (2^4 y)^3 &= (u-12)(u-4)(u+4)(u+12) \\ &= (u^2-12^2)(u^2-4^2) \\ &= (z-16)(z-144). \end{aligned} \tag{12}$$

Assume some prime p divides $z-144$ and $z-16$. Then it divides their difference, $128 = 2^7$. Thus,

$$z - 16 \ = \ 2^a v^3, \quad z - 144 \ = \ 2^b w^3. \tag{13}$$

Without loss of generality, we may take $0 \le a, b \le 2$, as if either is 3 or more, we may incorporate those factors into v or w. Thus,

$$2^{12} y^3 \ = \ 2^{a+b} v^3 w^3. \tag{14}$$

This implies that $a + b \equiv 0 \bmod 3$ (the number of powers of 2 on the LHS is a multiple of 3, and the factors of v^3 and w^3 give powers of 2 that are multiples of 3). There are thus three cases, $a = b = 0$, $a = 2$ and $b = 1$, and $a = 1$ and $b = 2$. As $z - 16 = 2^a v^3$ and $z - 144 = 2^b w^3$, in all cases we have

$$128 \ = \ 2^a v^3 - 2^b w^3. \tag{15}$$

We now show there are no solutions.

Case One: $a = b = 0$**.** This implies that

$$128 \ = \ v^3 - w^3. \tag{16}$$

From this, we deduce that

$$128 \ = \ (v - w)(v^2 + vw + w^2). \tag{17}$$

Clearly, $u, v < 2^3\sqrt{2}$ (ie, at most 11). Write $v = w + 2^c$ (as $(v-w)|128$, $v - w = 2^c$ for some $c \le 3$). Then

$$128 \ = \ 2^c(3w^2 + 3 \cdot 2^c w + 2^{2c}). \tag{18}$$

We have $c \in \{0, 1, 2, 3\}$. Subtracting gives

$$\begin{aligned} 2^{7-c} &= 3w^2 + 3 \cdot 2^c w + 2^{2c} \\ 2^{7-c} - 2^{2c} &= 3(w^2 + 2^c w). \end{aligned} \tag{19}$$

If $c = 3$, simple algebra yields $w^2 + 8w + 16 = 0$, or $w = 4$. Substituting yields $v = 4$, but then y is negative, contradicting x being a positive integer (better, if $v = 4$, then $z - 16 = 4^3$, or $z = u^2 = 80$, and this has no solutions in integers). Thus, $c \neq 3$. If $c = 0, 1, 2$, then $2^{7-c} - 2^{2c} = 127, 60, 16$ respectively. As 127 and 16 are not divisible by 3, the only possible solution is when $c = 1$. In this case we find $20 = w^2 + 4w$, which has no solution in positive integers.

Case Two: $a = 2, b = 1$**.** In this case, we have

$$z - 16 \ = \ 4v^3, \quad z - 144 \ = \ 2w^3. \tag{20}$$

As $z = u^2$, the first equation becomes

$$u^2 \ = \ 4v^3 + 16. \tag{21}$$

This is an elliptic curve. Changing variables by letting $u_2 = \frac{u}{4}$ and $v_2 = \frac{v}{4}$ gives the elliptic curve

$$E: \ u_2^2 \ = \ v_2^3 + 4^2 \cdot 16. \tag{22}$$

As $L(E, 1) \approx 1.11$, this curve has rank zero. Therefore, the Mordell-Weil group has no infinite part. Thus, all solutions are torsion points. Direct calculation gives the torsion group is $\mathbb{Z}/3\mathbb{Z}$, generated by the point $[0, 16]$. Thus, the torsion points are

$$\{[0, 16], [0, -16], [0]\}, \tag{23}$$

where $[0]$ is the additive identity. Thus, the solutions are $[u_2, v_2] = [0, \pm 16]$, which corresponds to $[u, v] = [0, \pm 64]$. However, $u = 0$ implies that $z = 0$, which yields our product is

$$(z - 16)(z - 144) \ = \ 16 \cdot 144 \ = \ 2^8 \cdot 3^3, \tag{24}$$

which is not a perfect cube. Thus, there are no solutions in this case.

Case Three: $a = 1, b = 2$**.** In this case, we have

$$z - 16 \ = \ 2v^3, \quad z - 144 \ = \ 4w^3. \tag{25}$$

As $z = u^2$, the first equation becomes

$$u^2 \ = \ 2v^3 + 16. \tag{26}$$

This is an elliptic curve. Changing variables by letting $u_2 = \frac{u}{2}$ and $v_2 = \frac{v}{2}$ gives the elliptic curve

$$E: \ u_2^2 \ = \ v_2^3 + 2^2 \cdot 16. \tag{27}$$

As $L(E,1) \approx .70$, this curve has rank zero. Therefore, the Mordell-Weil group has no infinite part. Thus, all solutions are torsion points. Direct calculation gives the torsion group is $\mathbb{Z}/6\mathbb{Z}$, generated by the point $[8,24]$. Thus, the torsion points are

$$\{[8,24],\ [0,8],\ [-4,0],\ [0,-8],\ [8,-24],\ [0]\}, \tag{28}$$

where $[0]$ is the additive identity. This implies that the rational solutions to the original elliptic curve are

$$[u,v] \ \in\ \{[16,48],\ [0,16],\ [-8,0],\ [0,-16],\ [16,-48]\}. \tag{29}$$

If $u = 0$ then $z = 0$, and we have seen in the previous case that there is no solution in this case. If $v = 0$, then the product $x(x+1)(x+2)(x+3) = 0$, which cannot happen for x a positive integer. We are reduced to checking $[u,v] \in \{[16,48],[16,-48]\}$. We now use the other equation, namely that $z - 144 = 4w^3$. As $z = u^2$, for both candidates we have $z = 16^2 = 256$, which yields

$$166 - 144 \ =\ 4w^3, \tag{30}$$

or

$$w^3 \ =\ 28. \tag{31}$$

As this equation has no solution in positive integers, this completes the proof that there are no solutions.

3 Alternate Thoughts on Being a Cube: I

We want

$$x(x+1)(x+2)(x+3) \ =\ y^3. \tag{32}$$

Letting $u = x - 1$ we may re-write the above as

$$(u-1)u(u+1)(u+2) \ =\ y^3. \tag{33}$$

The only divisors any of the four factors can have in common are 2 and 3.

Assume that 3 divides at most one of the factors. Thus, 3 divides either u or $u+1$. Split the multiplication into two parts, $(u-1)(u+1)$ and $u(u+2)$. All the factors of 2 occur in either the first multiplication or the second, but not both. As we are assuming 3 divides u or $u + 1$, this implies that each of the two multiplications must be a perfect cube. In particular, we have

$$(u-1)(u+1) \; = \; w^3. \tag{34}$$

This simplifies to

$$u^2 - w^3 \; = \; 1. \tag{35}$$

This is the Catalan Equation, which is now known to have just one solution, namely $u = 3$ and $w = 2$. Substituting in for u gives

$$(3-1)(3)(3+1)(3+2) \; = \; 120 \; = \; 2^3 \cdot 3 \cdot 5, \tag{36}$$

which is not a perfect square.

We are left with the case when $3|u$ and $3|(u+2)$. Clearly $2|u(u+1)$. If, however, 4 does not divide $u(u+1)$, then we must have

$$u(u+1) \; = \; 2w^3, \quad (u-1)(u+2) \; = \; 2^2v^3. \tag{37}$$

Multiplying the first equation by 4 gives

$$(2u)(2u+2) \; = \; (2w)^3. \tag{38}$$

Let $z = 2u + 1$. Then the above equation becomes

$$(z-1)(z+1) \; = \; (2w)^3, \tag{39}$$

which may be re-written as

$$z^2 - (2w)^3 \; = \; 1. \tag{40}$$

We again obtain the Catalan equation, which now has the unique solution $z = 3$, $w = 1$. If $z = 3$ then $u = 1$, and $(u-1)u(u+1)(u+2) = 0$, implying there are no solutions.

Thus, we are left with the case when $3|u$, $3|(u+2)$, and $4|u(u+1)$. We could use elliptic curve arguments again. If $(u-1)(u+1) \equiv 9 \bmod 27$, we would have

$$(u-1)(u+1) \; = \; 9w^3. \tag{41}$$

This leads to the elliptic curve

$$u^2 \ = \ 9w^3 + 1. \tag{42}$$

Letting $u_2 = \frac{u}{2}$ and $w_2 = \frac{w}{2}$ we obtain the elliptic curve

$$E: \ u_2^2 \ = \ w_2^3 + 81. \tag{43}$$

As $L(E, 1) \approx 2.02$, this curve has rank 0, and the only rational solutions are the torsion points. Direct calculation gives the torsion group is $\mathbb{Z}/6\mathbb{Z}$, generated by $[0, 9]$. Further computation should yield none of these give valid solutions to the original equation. Unfortunately, if $(u-1)(u+1) \equiv 3 \bmod 27$, we obtain a rank 2 elliptic curve, which is a little harder to analyze. Fortunately, if this is the case than instead of looking at $(u-1)(u+1)$, we can look at $u(u+2)$, which is equivalent to $9 \bmod 27$. Letting $z = u - 1$, this gives us

$$(z-1)(z+1) \ = \ 9v^3, \tag{44}$$

and this is the same equation as before. It will also have zero rank, and torsion group $\mathbb{Z}/6\mathbb{Z}$ generated by $[0, 9]$. Direct calculation will finish the proof.

4 Alternate Thoughts on Being a Cube: II

Case Two: $a = 2, b = 1$. In this case, we have

$$\begin{aligned} 128 &= 2^2v^3 - w^3 \\ 64 &= 2v^3 - w^3 \\ 64 + w^3 &= 2v^3. \end{aligned} \tag{45}$$

We must have w even, say $w = 2w_1$. This implies

$$64 + 8w_1^3 \quad = \quad 2v^3. \tag{46}$$

As 8 divides the LHS, we must have $8|2v^3$, so $v = 2v_1$, yielding

$$\begin{aligned} 64 + 8w_1^3 &= 16v_1^3 \\ 8 + w_1^3 &= 2v_1^3. \end{aligned} \tag{47}$$

Again, we must have $w_1 = 2w_2$, which then implies $v_1 = 2v_2$, giving

$$\begin{aligned} 8 + 8w_2^3 &= 16v_2^3 \\ 1 + w_2^3 &= 2v_2^3 \\ (w_2+1)(w_2^2 - w + 1) &= 2v_2^3. \end{aligned} \tag{48}$$

If a prime p divides both factors on the LHS, then it divides w_2+1 and $w_2^2-w+1 = w_2(w_2+1) - (2w_2 - 1)$. Thus, it divides both $w_2 + 1$ and $2w_2 - 1$, implying it divides both $w_2 + 1$ and $w_2 - 2$, implying it divides 3. We also have w_2 is odd. Thus,

$$\begin{aligned} w_2 + 1 &= 2 \cdot 3^c \alpha^3 \\ w_2^2 - w_2 + 1 &= 3^d \beta^3, \end{aligned} \tag{49}$$

where we may assume $0 \leq \alpha, \beta \leq 2$ and $\alpha + \beta \equiv 0 \bmod 3$.

We know, however, that $3|x(x+1)(x+2)(x+3)$; therefore, as it is a perfect cube, we must have $27|x(x+1)(x+2)(x+3)$. We showed earlier that the only common factor of v and w is 2; therefore, the only common factor of v_2 and w_2 is 2. Thus, either $27|v_2^3$ or $27|w_2^3$. In the first case, we have $1 + w_2^3 \equiv 0 \bmod 3$, which implies $w_2 \equiv 2 \bmod 3$. Since w_2 is clearly odd, this forces $w_2 \equiv 5 \bmod 6$. Using $1 + w_2^3 \equiv 0 \bmod 27$ yields $w_2 \equiv 17 \bmod 27$.

5 $x(x+1)(x+2)(x+3)$ is never a perfect power

We use the following result:

Theorem 5.1 (Mihailescu 2002). *Let $a, b \in \mathbb{Z}$ and $n, m \geq 2$ positive integers. Consider the equation*

$$a^n - b^m = \pm 1. \tag{50}$$

The only solution are $3^2 - 2^3 = 1$ and $1^n - 0^m = 1$.

Consider

$$x(x+1)(x+2)(x+3 = y^3. \tag{51}$$

We can re-group the factors and obtain

$$x(x+3) \cdot (x+1)(x+2) = \left(x^2 + 3x\right) \cdot (x^2 + 3x + 2) = y^3. \tag{52}$$

Letting $z = x^2 + 3x + 1$, we find that

$$(z - 1)(z + 1) \ = \ y^3. \tag{53}$$

We may re-write this as

$$z^2 - y^3 \ = \ 1. \tag{54}$$

The only solution is $z = 3, y = 2$, and this does not correspond to x a positive integer.

We now consider the obvious generalization to showing that $x(x+1)(x+2)(x+3)$ is never a perfect power. The only change in the previous argument is that we now have y^m instead of y^3 for some positive integer $m \geq 2$. We again obtain

$$z^2 - y^m \ = \ 1, \tag{55}$$

and again $z = x^2 + 3x + 1 = 3$, which has no solution. Note this also handles the case $m = 2$ (ie, $x(x+1)(x+2)(x+3)$ is never a square). This immediately gives

$$z^2 - 1 \ = \ y^2 \tag{56}$$

or equivalently

$$z^2 \ = \ y^2 + 1, \tag{57}$$

and there are no adjacent perfect squares other than 0 and 1; note $z = 0$ yields a non-integral x.