Some Results Concerning Terminating Triangle Sequences

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Abstract

After a brief review of the standard triangle iteration, we will explore some properties of terminating triangle sequences. Our first result will be motivated by the geometry of the iteration. We prove that, for any real number $\alpha \in (0,1)$ and any positive integer n, it is possible to find a corresponding β with $0 \leq \beta \leq \alpha$ such that the pair (α, β) terminates after n iterations. Our next results focus on two classes of quadratic irrationals that give particularly regular terminating sequences.

1 Introduction

This paper will start with an overview of the standard triangle iteration as outlined in work by Garrity [1]. We will begin with the geometrical interpretation of triangle sequences. We define an iteration T on the triangle

$$\triangle = \{(x,y) : 1 \geq x \geq y > 0\}.$$

This triangle is partitioned into an infinite set of disjoint subtriangles

$$\triangle_k = \{(x,y) \in \triangle : 1 - x - ky \geq 0 > 1 - x - (k+1)y\},$$

where k is any nonnegative integer.

Given $(\alpha, \beta) \in \triangle_k$, define the map $T : \triangle \to \triangle \cup \{(x,0) : 0 \leq x \leq 1\}$ by

$$T(\alpha,\beta) = \left(\frac{\beta}{\alpha}, \frac{1-\alpha-k\beta}{\alpha}\right).$$

The triangle sequence is recovered from this iteration by keeping track of the number of the triangle that the point is mapped into at each step. In other words, if $T^{k-1}(\alpha,\beta) \in \triangle_{a_k}$, then the point (α,β) will have the triangle sequence $(a_1, a_2, \dots)$.

This paper deals with properties of terminating triangle sequences. A triangle sequence for (α, β) is said to terminate after n steps if $T^n(\alpha,\beta)$ is on the line segment bounded by $(0,0)$ and $(1,0)$. Thus, we can understand the set of points with a particular finite triangle sequence as the appropriate preimage of that segment bounded by $(0,0)$ and $(1,0)$. [2]

In section 2 of this paper, we use this geometric notion of termination to prove that for any α in the interval $(0,1)$ there exists a β such that $(\alpha,\beta) \in \triangle$ and $T^n(\alpha,\beta)$ lands on the segment bounded by $(0,0)$ and $(1,0)$. In other words, for every α one can find a β such that the triangle sequence for the pair (α,β) terminates after n terms.

Sections 3 and 4 explore two separate classes of quadratic irrationals α such that the triangle sequence for (α,α^2) terminates with a regular predictable pattern. For both of these classes, we define α in a similar way. This relationship and the palindromic-like triangle sequences strongly suggest a broader class of quadratic irrational forms which also have similar structure to their triangle sequences.

2 Existence of Terminating Sequences of Length n

Theorem 1 *For any $\alpha \in (0,1)$ and any $n \in \mathbb{Z}^+, \exists \beta \in (0,\alpha]$ such that the sequence for (α,β) terminates after n terms.*

Proof:

Let $I = (0,1) \times \{0\}$. The triangle sequence for the point (α,β) terminates after n terms if $T^n(\alpha,\beta) \in I$.

Notation : T_k^{-1} is the inverse of the map T restricted to $\triangle_k$,
$T_k^{-1}(x,y) = (\frac{1}{kx+y+1}, \frac{x}{kx+y+1})$. $T^{-n}_{k_n\ldots k_1} = T^{-1}_{k_n}(T^{-(n-1)}_{k_{n-1}\ldots k_1})$.

If the vertical line $x=\alpha$ intersects a preimage $T^{-n}_{k_n\ldots k_1}(I)$ for some sequence $k_n, k_{n-1}\ldots k_1$ of nonnegative integers, then the triangle sequence for the point of intersection is $k_n, k_{n-1}\ldots k_1$. So we must show that for any $n \in \mathbb{Z}^+$, the line $x=\alpha$ intersects some $T^{-n}_{k_n\ldots k_1}(I)$.

The preimage $T^{-n}_{k_n\ldots k_1}(I)$ is the line segment between $T^{-n}_{k_n\ldots k_1}(0,0)$ and $T^{-n}_{k_n\ldots k_1}(1,0)$. For $n=1$, we have $T_k^{-1}(0,0) = (\frac{1}{0+0+1}, \frac{0}{0+0+1}) = (1,0)$ and
$T_k^{-1}(1,0) = (\frac{1}{k+1}, \frac{1}{k+1})$. So, for any $\alpha \in (\frac{1}{k+1}, 1), x = \alpha$ intersects $T_k^{-1}(I)$.

Now suppose $x=\alpha$ intersects $T^{-n}_{k_n\ldots k_1}(I)$. We shall show that for some integer $l \geq 0,\ x=\alpha$ intersects $T^{-(n+1)}_{k_n\ldots k_1,l}(I)$. First, note that:

$$\begin{aligned}
T^{-n}_{x_n\ldots x_1}(0,0) &= T^{-(n-1)}_{x_n\ldots x_2}(T^{-1}_{x_1}(0,0)) \\
&= T^{-(n-1)}_{x_n\ldots x_2}(1,0) \\
&= T^{-(n-1)}_{x_n\ldots x_2}(T^{-1}_{y}(0,0)) \\
&= T^{-n}_{x_n\ldots x_2,y}(0,0).
\end{aligned}$$

In particular, $T^{-n}_{k_n\ldots k_2,k_1}(0,0) = T^{-n}_{k_n\ldots k_2,k_1+1}(0,0)$, and $T^{-(n+1)}_{k_n\ldots k_1,0}(0,0) = T^{-(n+1)}_{k_n\ldots k_1,l}(0,0)$ for all l. Furthermore, $T^{-(n+1)}_{k_n\ldots k_1,0}(0,0) = T^{-n}_{k_n\ldots k_1}(T_0^{-1}(0,0)) = T^{-n}_{k_n\ldots k_1}(1,0)$.

Now,

$$\begin{aligned}
T^{-(n+1)}_{k_n\ldots k_1,h}(1,0) &= T^{-(n-1)}_{k_n\ldots k_2}(T^{-1}_{k_1}(T^{-1}_h(1,0))) \\
&= T^{-(n-1)}_{k_n\ldots k_2}(T^{-1}_{k_1}(\frac{1}{h+1},\frac{1}{h+1})) \\
&= T^{-(n-1)}_{k_n\ldots k_2}(\frac{h+1}{k_1+1+h+1},\frac{1}{k_1+1+h+1}) \\
&= T^{-(n-1)}_{k_n\ldots k_2}(\frac{1}{\frac{k_1+1}{h+1}+0+1},\frac{\frac{1}{h+1}}{\frac{k_1+1}{h+1}+0+1}) \\
&= T^{-(n-1)}_{k_n\ldots k_2}(T^{-1}_{k_1+1}(\frac{1}{h+1},0)) \\
&= T^{-n}_{k_n\ldots k_2,k_1+1}(\frac{1}{h+1},0).
\end{aligned}$$

So, if $x=\alpha$ intersects $T^{-n}_{k_n\ldots k_1}(I)$, then there is a positive integer l such that α is between the first coordinates of $T^{-n}_{k_n\ldots k_1}(1,0)$ and $T^{-n}_{k_n\ldots k_1+1}(\frac{1}{l+1},0)$. But these are the endpoints of $T^{-(n+1)}_{k_n\ldots k_1,l}(I)$, so $x=\alpha$ intersects this segment.

$\square$

3 Palindrome Class Number One

In this section of the paper we will define a class of quadratic irrational pairs (α,β) which have a regular terminating triangle sequence. For this class of ordered pairs α will be completely determined by a positive integer n and β will be completely determined by α.

Theorem 2 *Given any (α_1,β_1) pair of the form*

$$\begin{aligned}
\alpha_1 &= \sqrt{n^2-1}-n+1, \text{ where } n \text{ is any positive integer} \\
\beta_1 &= \alpha^2
\end{aligned}$$

then $\triangle_{(\alpha_1,\beta_1)}$ is:

$$(0,0,(2n-3),0,1,(2n-4),0,2,(2n-5),\ldots,0,(2n-4),1,1)$$

Proof:

The Base Cases:

We know that,

$$\begin{aligned}
a_1 &= \left\lfloor \frac{1-\alpha_1}{\beta_1} \right\rfloor \\
&= \left\lfloor \frac{1-(\sqrt{n^2-1}-n+1)}{(2n-2)(n-\sqrt{n^2-1})} \right\rfloor \\
&= \left\lfloor \frac{n-\sqrt{n^2-1}}{(2n-2)(n-\sqrt{n^2-1})} \right\rfloor \\
&= \left\lfloor \frac{1}{2n-2} \right\rfloor .
\end{aligned}$$

Since $n \geq 2$ we know $2n-2 \geq 1$. This implies that $\left\lfloor \frac{1}{2n-2} \right\rfloor = 0$. Therefore, $a_1 = 0$.

Additionally, we know that,

$$\begin{aligned}
(\alpha_2, \beta 2) &= T(1-n+\sqrt{n^2-1},\ (2n-2)(n-\sqrt{n^2-1})) \\
&= (1-n+\sqrt{n^2-1}, \frac{n-\sqrt{n^2-1}}{1-n+\sqrt{n^2-1}})
\end{aligned}$$

Therefore,

$$a_2 = \left\lfloor \frac{1-\alpha_2}{\beta_2} \right\rfloor = \left\lfloor \frac{1-(1-n+\sqrt{n^2-1})}{\frac{n-\sqrt{n^2-1}}{1-n+\sqrt{n^2-1}}} \right\rfloor = \left\lfloor 1-n+\sqrt{n^2-1} \right\rfloor .$$

We will show that $0 \leq 1-n+\sqrt{n^2-1} < 1$. For all $n \geq 2$,

$$\begin{aligned}
2n &\geq 2 \\
-2n+1 &\leq -1 \\
n^2-2n+1 &\leq n^2-1 \\
n-1 &\leq \sqrt{n^2-1} \\
0 &\leq 1-n+\sqrt{n^2-1}
\end{aligned}$$

and

$$\begin{aligned}
n^2-1 &< n^2 \\
\sqrt{n^2-1} &< n \\
\sqrt{n^2-1}-n &< 0 \\
1-n+\sqrt{n^2-1} &< 1
\end{aligned}$$

Therefore, $a_2 = 0$.

Similarily, we have,

$$\begin{aligned}
(\alpha_3, \beta_3) &= T\left(1-n+\sqrt{n^2-1}, \frac{n-\sqrt{n^2-1}}{1-n+\sqrt{n^2-1}}\right) \\
&= \left(\frac{n-\sqrt{n^-1}}{(1-n+\sqrt{n^2-1})^2}, \frac{1-(1-n+\sqrt{n^2-1})}{1-n+\sqrt{n^2-1}}\right) \\
&= \left(\frac{n-\sqrt{n^-1}}{(2n-2)(n-\sqrt{n^2-1})}, \frac{n-\sqrt{n^2-1}}{1-n+\sqrt{n^2-1}}\right) \\
&= \left(\frac{1}{2n-2}, \frac{1-n+\sqrt{n^2-1}}{2n-2}\right).
\end{aligned}$$

Therefore,

$$a_3 = \left\lfloor \frac{1-\frac{1}{2n-2}}{\frac{1-n+\sqrt{n^2-1}}{2n-2}} \right\rfloor = \left\lfloor \frac{\frac{2n-3}{2n-2}}{\frac{1-n+\sqrt{n^2-1}}{2n-2}} \right\rfloor = \left\lfloor \frac{2n-3}{1-n+\sqrt{n^2-1}} \right\rfloor.$$

We will show that $2n-3 \leq \frac{2n-3}{1-n+\sqrt{n^2-1}} \leq 2n-2$. For all $n \geq 2$,

$$\begin{aligned}
n^2-1 &< n^2 \\
\sqrt{n^2-1} &< n \\
1+\sqrt{n^2-1} &< n+1 \\
1-n+\sqrt{n^2-1} &< 1 \\
\frac{2n-3}{1-n+\sqrt{n^2-1}} &> 2n-3
\end{aligned}$$

and

$$\begin{aligned}
4n &> 5 \\
4n+1 &< 8n-4 \\
4n^4-8n^3+4n+1 &< 4n^4-8n^3+8n-4 \\
4n^4-8n^3+4n+1 &< (n^2-1)(4n^2-8n+4) \\
\frac{(2n^2-2n-1)^2}{(2n-2)^2} &< n^2-1 \\
n-\frac{1}{2n-2} &< \sqrt{n^2-1} \\
1-\frac{1}{2n-2} &< 1-n+\sqrt{n^2-1} \\
\frac{2n-3}{1-n+\sqrt{n^2-1}} &< 2n-2.
\end{aligned}$$

Therefore, $a_3 = 2n-3$.

The Inductive Step:
An inequality that we will need later is:

$$\frac{1}{2n-1} > n - \sqrt{n^2-1} \tag{1}$$

Proof:

$$\begin{aligned}
1 &> -1 \\
4n^3 - 3n + 1 &> 4n^3 - 3n - 1 \\
(n+1)(4n^2 - 4n + 1) &> (4n^2 + 4n + 1)(n-1) \\
(n^2-1)(2n-1)^2 &> (2n+1)^2(n-1)^2 \\
\sqrt{n^2-1}(2n-1) &> (2n^2 - n - 1) \\
1 &> n(2n-1) - (2n-1)\sqrt{n^2-1} \\
\frac{1}{2n-1} &> n - \sqrt{n^2-1}
\end{aligned}$$

□

It is easier to deal with the calculations in this proof if we make a few substitutions along the way. Let

$$x = 1 - n + \sqrt{n^2-1}$$

and let

$$y = 2n - 2.$$

Note that $(1-x)y = x^2$. Then our results from the previous section are:

$\mathbf{k}$	$\mathbf{a_k}$	$\alpha_\mathbf{k}$	$\beta_\mathbf{k}$
1	0	x	$y(1-x)$
2	0	x	$\frac{1-x}{x}$
3	$y-1$	$\frac{1}{y}$	$\frac{x}{y}$

We will now inductively show that the sequences $k, a_k, \alpha_k, \beta_k$ continue in their respective patterns above. Let,

$\mathbf{k}$	$\mathbf{a_k}$	$\alpha_\mathbf{k}$	$\beta_\mathbf{k}$
c	0	x	$(y-r)(1-x)$
$c+1$	r	$\frac{x(y-r)}{y}$	$\frac{1-x}{x}$
$c+2$	$y-r-1$	$\frac{1}{y-r}$	$\frac{x}{y-r}$

be three consecutive terms in the sequence where $r \leq y-3$. Then, we know that,

$$\begin{aligned} T(\alpha_{c+2}, \beta_{c+2}) &= \left(x, \frac{1-\frac{1}{y-r}-(y-r-1)\frac{x}{y-r}}{\frac{1}{y-r}}\right) \\ &= (x, (y-r-1)(1-x)) \\ &= (\alpha_{c+3}, \beta_{c+3}). \end{aligned}$$

Therefore,

$$a_{c+3} = \left\lfloor \frac{1-x}{(y-r-1)(1-x)} \right\rfloor = \left\lfloor \frac{1}{y-r-1} \right\rfloor = 0.$$

Additionally, we know that,

$$\begin{aligned} T(\alpha_{c+3}, \beta_{c+3}) &= (\frac{(y-r-1)(1-x)}{x}, \frac{1-x}{x}) \\ &= (\frac{x(y-r-1)}{y}, \frac{1-x}{x}) \\ &= (\alpha_{c+4}, \beta_{c+4}). \end{aligned}$$

Therefore,

$$\begin{aligned} a_{c+4} &= \left\lfloor \frac{1-\frac{x(y-r-1)}{y}}{\frac{1-x}{x}} \right\rfloor \\ &= \left\lfloor \frac{\frac{(2n-2)-(2n-3-r)(1-n+\sqrt{n^2-1})}{2n-2}}{\frac{1-n-\sqrt{n^2-1}}{2n-2}} \right\rfloor \\ &= \left\lfloor \frac{2n-2-(2n-3-r)(1-n+\sqrt{n^2-1})}{1-n-\sqrt{n^2-1}} \right\rfloor \\ &= \left\lfloor \frac{2n-2}{1-n+\sqrt{n^-2}} - (2n-3-r) \right\rfloor \end{aligned}$$

We will show that $r+1 \leq \frac{2n-2}{1-n+\sqrt{n^-2}} - (2n-3-r) < r+2$. To get the left inequality, we observe that for all $n \geq 2$,

$$\begin{aligned} \sqrt{n^2-1} &< n \\ 1-n+\sqrt{n^2-1} &< 1 \\ 2n-2 &< \frac{2n-2}{1-n+\sqrt{n^2-1}} \\ r+1 &< \frac{2n-2}{1-n+\sqrt{n^2-1}} - (2n-3-r). \end{aligned}$$

To get the right inequality we recall Equation 1,

$$\begin{aligned}
\frac{1}{2n-1} &> n-\sqrt{n^2-1}\\
\frac{2n-2}{2n-1} &< 1-n+\sqrt{n^2-1}\\
\frac{2n-2}{1-n+\sqrt{n^2-1}} &< 2n-1\\
\frac{2n-2}{1-n+\sqrt{n^2-1}}-(2n-3-r) &< r+2
\end{aligned}$$

Therefore, $a_{c+4}=r+1$.

Similarily, we have,

$$\begin{aligned}
T(\alpha_{c+4},\beta_{c+4}) &= \left(\frac{\frac{1-x}{x}}{\frac{x(y-r-1)}{y}}, \frac{1-\frac{x(y-r-1)}{y}-(r+1)\frac{1-x}{x}}{\frac{x(y-r-1)}{y}}\right)\\
&= \left(\frac{y(1-x)}{x^2(y-r-1)}, \frac{y-(y-r-1)x-(r+1)x}{(y-r-1)x}\right)\\
&= \left(\frac{1}{y-r-1}, \frac{y-xy}{(y-r-1)x}\right)\\
&= \left(\frac{1}{y-r-1}, \frac{x}{y-r-1}\right)\\
&= (\alpha_{c+5},\beta_{c+5}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_{c+5} &= \left\lfloor\frac{1-\frac{1}{y-r-1}}{\frac{x}{y-r-1}}\right\rfloor\\
&= \left\lfloor\frac{y-r-2}{x}\right\rfloor\\
&= \left\lfloor\frac{2n-r-4}{1-n+\sqrt{n^2-1}}\right\rfloor
\end{aligned}$$

We will show that,

$$y-r-2=2n-r-4\leq\frac{2n-r-4}{1-n+\sqrt{n^2-1}}<y-r-1=2n-r-3.$$

To get the left inequality, we observe that for all $n\geq 2$,

$$\begin{aligned}
\sqrt{n^2-1} &< n\\
1-n+\sqrt{n^2-1} &< \frac{2n-r-4}{2n-r-4}\\
2n-r-4 &< \frac{2n-r-4}{1-n+\sqrt{n^2-1}}
\end{aligned}$$

To get the right inequality we recall Equation 1,

$$\begin{aligned}
\frac{1}{2n-1} &> n-\sqrt{n^2-1}\\
\frac{2n-2}{2n-1} &< 1-n+\sqrt{n^2-1}\\
\frac{2n-r-4}{2n-r-3} &< 1-n+\sqrt{n^2-1}\\
\frac{2n-r-4}{1-n+\sqrt{n^2-1}} &< 2n-r-3
\end{aligned}$$

Therefore, $a_{c+5} = y-r-2$.

Thus by induction, we have:

$$\begin{aligned}
a_k = 0 &\quad if \quad k \equiv 1 \pmod 3\\
a_k = \frac{k-2}{3} &\quad if \quad k \equiv 2 \pmod 3\\
a_k = y-1-\frac{k-3}{3} &\quad if \quad k \equiv 0 \pmod 3
\end{aligned}$$

It follows that after $2(2n-3)$ iterations, we will have $a_k = 0$, $\alpha_k = \frac{1}{2}$ and $\beta_k = \frac{x}{2}$. Then, $T(\alpha_k, \beta_k) = (x, 1-x)$ which lies on the line dividing triangles zero and one. Therefore, the triangle sequence will terminate with $a = 1$ after $4n-6$ terms.

□

4 Palindrome Class Number Two

In this section we will define another class of quadratic irrational ordered pairs which have a very regular triangle sequence. Once again, α is determined by a positive integer, n, and β is determined by α.

Theorem 3 *For any $n \in \mathbb{Z}^+$, if $\alpha = \frac{\sqrt{n^2+1}-n+1}{2}$ and $\beta = \alpha^2 = \frac{n^2-n+1+(1-n)\sqrt{n^2+1}}{2}$, then the triangle sequence for (α, β) terminates, with the following pattern:*

$$\underbrace{1,1,1,n-5}_{p=0},\underbrace{1,1,5,n-9}_{p=1},\ldots,\underbrace{1,1,4p+1,n-4p-5}_{p},\ldots\begin{cases}0,0,1 & n=1(\text{mod }4)\\ 1 & n=2(\text{mod }4)\\ 1,3 & n=3(\text{mod }4)\\ 1,2 & n=0(\text{mod }4)\end{cases}$$

Proof:

Let $(\alpha_k, \beta_k) = T^{k-1}(\alpha, \beta)$, so the k^{th} term of the triangle sequence is $a_k = \left\lfloor \frac{1-\alpha_k}{\beta_k} \right\rfloor$. Now suppose that for some integer $p \geq 0$,

$$(\alpha_{4p+1}, \beta_{4p+1}) = \left(\frac{1 - n + \sqrt{n^2+1}}{2}, \frac{(n^2 - (4p+1)n + 1) + (4p+1-n)\sqrt{n^2+1}}{2} \right).$$

Note that for $p = 0$, this gives the original pair (α, β).

We will consider two cases. First we will show that when $n \geq 4p + 5$, the next four terms of the triangle sequence are $1, 1, 4p + 1, n - 4p - 5$, and that $(\alpha_{4(p+1)+1}, \beta_{4(p+1)+1})$ is of the same form as $(\alpha_{4p+1}, \beta_{4p+1})$, i.e.

$$(\alpha_{4p+5}, \beta_{4p+5}) = \left(\frac{1 - n + \sqrt{n^2+1}}{2}, \frac{(n^2 - (4(p+1)+1)n + 1) + (4(p+1)+1-n)\sqrt{n^2+1}}{2} \right).$$

Then we will show that for $n < 4p + 5$, the sequence terminates within three terms, with the patterns given above.

Case I: $n \geq 4p + 5$

First we must show that $a_{4p+1} = 1$.

But $a_{4p+1} = \left\lfloor \frac{1-\alpha_{4p+1}}{\beta_{4p+1}} \right\rfloor$, and $\frac{1-\alpha_{4p+1}}{\beta_{4p+1}} = \frac{1+n-\sqrt{n^2+1}}{(n^2-(4p+1)n+1)+(4p+1-n)\sqrt{n^2+1}}$. So we must show that $1 < \frac{1-\alpha_{4p+1}}{\beta_{4p+1}} < 2$. But we know that $n^2 + 1 > n^2$, so we have:

$$\begin{aligned}
[n - (4p+2)]^2(n^2+1) &> [n - (4p+2)]^2 n^2 \\
(n - (4p+2))\sqrt{n^2+1} &> n^2 - (4p+2)n \\
1 + n - \sqrt{n^2+1} &> n^2 - (4p+1)n + 1 + (4p+1-n)\sqrt{n^2+1} \\
\frac{1 - \alpha_{4p+1}}{\beta_{4p+1}} &> 1
\end{aligned}$$

and

$$\begin{aligned}
4n^4 + 4n^2 &< 4n^4 + 4n^2 + 1 \\
2n\sqrt{n^2+1} &< 2n^2 + 1 \\
\sqrt{n^2+1} - n &< \frac{1}{2n}
\end{aligned}$$

Since $n \geq 4p + 5$, we know that $2n > 8p + 3$, so $2n > 2n - (8p + 3) > 0$. Therefore:

$$\begin{aligned}
\sqrt{n^2+1} - n &< \frac{1}{2n - (8p+3)} \\
(2n - (8p+3))(\sqrt{n^2+1} - n) &< 1 \\
(2n - (8p+3))\sqrt{n^2+1} &< 2n^2 - (8p+3)n + 1
\end{aligned}$$

Adding $1+n+(8p+2-2n)\sqrt{n^2+1}$ to both sides yields:

$$\begin{aligned}
1+n-\sqrt{n^2+1} &< 2n^2-2(4p+1)n+2+(8p+2-2n)\sqrt{n^2+1} \\
\frac{1-\alpha_{4p+1}}{\beta_{4p+1}} &< 2
\end{aligned}$$

So $a_{4p+1}=1$, as desired.

Now we find α_{4p+2} and β_{4p+2}:

$$\begin{aligned}
(\alpha_{4p+2},\beta_{4p+2}) &= \left(\frac{\beta_{4p+1}}{\alpha_{4p+1}}, \frac{1-\alpha_{4p+1}-\beta_{4p+1}}{\alpha_{4p+1}}\right) \\
&= \left(\frac{(n^2-(4p+1)n+1)+(4p+1-n)\sqrt{n^2+1}}{1-n+\sqrt{n^2+1}},\right. \\
&\qquad \left.\frac{2-(1-n)-\sqrt{n^2+1}-(n^2-(4p+1)n+1)-(4p+1-n)\sqrt{n^2+1}}{1-n+\sqrt{n^2+1}}\right)
\end{aligned}$$

Multiplying numerator and denominator by $1-n-\sqrt{n^2+1}$ yields:

$$\begin{aligned}
= &\left(\frac{(n^2-(4p+1)n+1)(1-n)-(4p+1-n)(n^2+1)}{-2n}+\right. \\
&\quad \frac{[(4p+1-n)(1-n)-(n^2-(4p+1)n+1)]\sqrt{n^2+1}}{-2n}, \\
&\left.\frac{((4p+2)n-n^2)(1-n)+(4p+2-n)(n^2+1)-[(4p+2-n)(1-n)+(4p+2)n-n^2]\sqrt{n^2+1}}{-2n}\right)
\end{aligned}$$

Expanding and simplifying, we have:

$$\begin{aligned}
&= \left(\frac{-n^2+(4p+1)n+4p+(n-4p)\sqrt{n^2+1}}{2n}, \frac{n^2-(4p+1)n-(4p+2)-(n-(4p+2))\sqrt{n^2+1}}{2n}\right) \\
&= \left(\frac{(n-4p)(1-n+\sqrt{n^2+1})}{2n}, \frac{(n-4p-2)(1+n-\sqrt{n^2+1})}{2n}\right).
\end{aligned}$$

To calculate $a_{4p+2} = \left\lfloor \frac{1-\alpha_{4p+2}}{\beta_{4p+2}} \right\rfloor$, note that

$$\frac{1-\alpha_{4p+2}}{\beta_{4p+2}} = \frac{n^2+(1-4p)n-4p-(n-4p)\sqrt{n^2+1}}{n^2-(4p+1)n-(4p+2)-(n-(4p+2))\sqrt{n^2+1}}.$$

Since $n+1 > \sqrt{n^2+1}$, we have:

$$\begin{aligned}
2(n+1) &> 2\sqrt{n^2+1} \\
n^2-(4p-1)n-4p-(n-4p)\sqrt{n^2+1} &> n^2-(4p+1)n-(4p+2)-(n-(4p+2))\sqrt{n^2+1} \\
\frac{1-\alpha_{4p+2}}{\beta_{4p+2}} &> 1.
\end{aligned}$$

Also, since $n > 4p+4$ and $\sqrt{n^2+1} < n+1$,

$$\begin{aligned}
(n-(4p+4))\sqrt{n^2+1} &< (n-(4p+4))(n+1) = n^2-(4p+3)n-(4p+4) \\
n^2+(1-4p)n-4p-(n-4p)\sqrt{n^2+1} &< 2n^2-2(4p+1)n-2(4p+2)+(-2n+8p+4)\sqrt{n^2+1} \\
\frac{1-\alpha_{4p+2}}{\beta_{4p+2}} &< 2.
\end{aligned}$$

So $a_{4p+2} = 1$, as desired.

Now:

$$\begin{aligned}
(\alpha_{4p+3}, \beta_{4p+3}) &= \left(\frac{\beta_{4p+2}}{\alpha_{4p+2}}, \frac{1-\alpha_{4p+2}-\beta_{4p+2}}{\alpha_{4p+2}} \right) \\
&= \left(\frac{n^2-(4p+1)n-(4p+2)-(n-(4p+2))\sqrt{n^2+1}}{-n^2+(4p+1)n+4p+(n-4p)\sqrt{n^2+1}}, \right. \\
&\qquad \left. \frac{2n+2-2\sqrt{n^2+1}}{-n^2+(4p+1)n+4p+(n-4p)\sqrt{n^2+1}} \right)
\end{aligned}$$

Multiplying numerators and denominators by the conjugate of the denominator and simplifying yields:

$$\begin{aligned}
&= \left(\frac{2(4p+1)n^2-2(4p+1)(4p+2)n-(2n^2-2(4p+2)n)\sqrt{n^2+1}}{-2n^3+32p^2n+16pn}, \right. \\
&\qquad \left. \frac{4(4p+1)n-4n\sqrt{n^2+1}}{-2n^3+32p^2n+16pn} \right) \\
&= \left(\frac{-(4p+1)(n-(4p+2))+(n-(4p+2))\sqrt{n^2+1}}{n^2-4p(4p+2)}, \frac{-2(4p+1)+2\sqrt{n^2+1}}{n^2-4p(4p+2)} \right).
\end{aligned}$$

Now,

$$a_{4p+3} = \left\lfloor \frac{1-\alpha_{4p+3}}{\beta_{4p+3}} \right\rfloor$$

and

$$\frac{1-\alpha_{4p+3}}{\beta_{4p+3}} = \frac{n^2+(4p+1)n-(8p+1)(4p+2)-(n-(4p+2))\sqrt{n^2+1}}{-2(4p+1)+2\sqrt{n^2+1}}.$$

Since $n > 4p+2$, we have:

$$\begin{aligned}
(4p+1)n+(4p+2) &< (4p+2)n \\
n^2+(4p+1)n+(4p+2) &< (n+4p+2)n
\end{aligned}$$

But $n < \sqrt{n^2+1}$, so:

$$\begin{aligned}
n^2+(4p+1)n+(4p+2) &< (n+4p+2)\sqrt{n^2+1} \\
n^2+(4p+1)n-(8p+1)(4p+2)- & \\
(n-(4p+2))\sqrt{n^2+1} &< -2(4p+1)(4p+2)+2(4p+2)\sqrt{n^2+1} \\
\frac{1-\alpha_{4p+3}}{\beta_{4p+3}} &< 4p+2.
\end{aligned}$$

We also know that $n^2 > 4p(4p+2)$, so:

$$\begin{aligned}
2n^3 &> 8p(4p+2) \\
2n^3-8p(4p+2)n &> 0 \\
n^4+2(4p+1)n^3 & \\
+(4p+1)^2n^2-8pn^n-8p(4p+1)n+16p^2 &> n^4+8pn^3+(16p^2+1)n^2+8pn+16p^2 \\
(n^2+(4p+1)n-4p)^2 &> (n+4p)^2(n^2+1)
\end{aligned}$$

Taking square roots and adding $-2(4p+1)^2-(n-(4p+2))\sqrt{n^2+1}$, we get:

$$\begin{aligned}
n^2-4p(4p+2)+(4p+1)n-(4p+1)(4p+2)- & \\
(n-(4p+2))\sqrt{n^2+1} &> -2(4p+1)^2+2(4p+1)\sqrt{n^2+1} \\
\frac{1-\alpha_{4p+3}}{\beta_{4p+3}} &> 4p+1.
\end{aligned}$$

So $a_{4p+3} = 4p+1$, as desired.

Now we calculate α_{4p+4} and β_{4p+4}.

$$\begin{aligned}
(\alpha_{4p+4}, \beta_{4p+4}) &= \left(\frac{\beta_{4p+3}}{\alpha_{4p+3}}, \frac{1-\alpha_{4p+3}-(4p+1)\beta_{4p+3}}{\alpha_{4p+3}}\right) \\
&= \left(\frac{2(-(4p+1)+\sqrt{n^2+1})}{(n-4p-2)(-(4p+1)+\sqrt{n^2+1})},\right. \\
&\quad \frac{n^2-16p^2-8p+(4p+1)n-(4p+1)(4p+2)}{(n-4p-2)(-(4p+1)+\sqrt{n^2+1})} \\
&\quad \left.\frac{-(n-4p-2)\sqrt{n^2+1}+2(4p+1)^2-2(4p+1)\sqrt{n^2+1}}{(n-4p-2)(-(4p+1)+\sqrt{n^2+1})}\right) \\
&= \left(\frac{2}{n-4p-2}, \frac{n^2+(4p+1)n-4p-(n+4p)\sqrt{n^2+1}}{(n-4p-2)(-(4p+1)+\sqrt{n^2+1}}\right) \\
&= \left(\frac{2}{n-4p-2}, \frac{n^3-n^2-4p(4p+2)n+4p(4p+2)-(n^2-4p(4p+2))\sqrt{n^2+1}}{(n-4p-2)(4p(4p+2)-n^2}\right) \\
&= \left(\frac{2}{n-4p-2}, \frac{n-1+\sqrt{n^2+1}}{n-4p-2}\right)
\end{aligned}$$

Now for a_{4p+4}, we have:

$$\frac{1-\alpha_{4p+4}}{\beta_{4p+4}} = \frac{n-4p-4}{n-1+\sqrt{n^2+1}} < n-4p-4,$$

since $n-1+\sqrt{n^2+1} > 1$.
And, if $n = 4p+5$, then

$$\frac{1-\alpha_{4p+4}}{\beta_{4p+4}} = \frac{1}{n-1+\sqrt{n^2+1}} > 0 = n-4p-5.$$

For $n > 4p+5$, we have:

$$\begin{aligned}
n^4+2n^2+1 &> n^4+n^2 \\
n^2+1 &> n\sqrt{n^2+1} \\
\frac{1}{n} &> \sqrt{n^2+1}-n
\end{aligned}$$

But $n > n-4p-5 > 0$, so:

$$\begin{aligned}
\frac{1}{n-4p-5} &> \sqrt{n^2+1}-n \\
1+\frac{1}{n-4p-5} &> 1-n+\sqrt{n^2+1} \\
\frac{n-4p-4}{n-4p-5} &> 1-n+\sqrt{n^2+1} \\
\frac{1-\alpha_{4p+4}}{\beta_{4p+4}} = \frac{n-4p-4}{1-n+\sqrt{n^2+1}} &> n-4p-5
\end{aligned}$$

so $a_{4p+4} = n-4p-5$.

Finally,

$$\begin{aligned}
(\alpha_{4p+5}, \beta_{4p+5}) &= \left(\frac{\beta_{4p+4}}{\alpha_{4p+4}}, \frac{1-\alpha_{4p+4}-(n-4p-5)\beta_{4p+4}}{\alpha_{4p+4}}\right) \\
&= \left(\frac{1-n+\sqrt{n^2+1}}{2}, \frac{n-4p-2-2-(n-4p-5)(1-n+\sqrt{n^2+1})}{2}\right) \\
&= \left(\frac{1-n+\sqrt{n^2+1}}{2}, \frac{n^2-(4(p+1)+1)n+1+(4(p+1)+1-n)\sqrt{n^2+1}}{2}\right),
\end{aligned}$$

as desired. Thus for any n, as p increases, the pattern $1, 1, 4p+1, n-4p-5$ continues to repeat in the triangle sequence as long as $n \geq 4p+5$.

Case II: Now we consider the case where $n < 4p+5$. We must address four cases, where $n = 4p+1, 4p+2, 4p+3,$ and$4p+4$. Note that if $\frac{1-\alpha_k}{\beta_k}$ is an integer, then the sequence terminates after a_k, since $\beta_{k+1} = 0$.

Case 1: $n = 4p+1$

We will show that the sequence terminates with the terms $a_{4p+1} = 0, a_{4p+2} = 0, a_{4p+3} = 1$.

For $n = 4p+1$, $(\alpha_{4p+1}, \beta_{4p+1}) = (\frac{1-n+\sqrt{n^2+1}}{2}, \frac{1}{2})$.

So

$$a_{4p+1} = \left\lfloor \frac{1-\alpha_{4p+1}}{\beta_{4p+1}} \right\rfloor = \lfloor 1+n-\sqrt{n^2+1} \rfloor.$$

But we have $n < \sqrt{n^2+1}$, so

$$1+n-\sqrt{n^2+1} < 1.$$

And $1+n > \sqrt{n^2+1}$, so

$$1+n-\sqrt{n^2+1} > 0.$$

Therefore, $a_{4p+1} = 0$.

Then we have:

$$\begin{aligned}
(\alpha_{4p+2}, \beta_{4p+2}) &= \left(\frac{\beta_{4p+1}}{\alpha_{4p+1}}, \frac{1-\alpha_{4p+1}}{\alpha_{4p+1}}\right) \\
&= \left(\frac{1}{1-n+\sqrt{n^2+1}}, \frac{1+n-\sqrt{n^2+1}}{1-n+\sqrt{n^2+1}}\right) \\
&= \left(\frac{1-n-\sqrt{n^2+1}}{-2n}, \frac{(1-n^2)+(n^2+1)-(1+n+1-n)\sqrt{n^2+1}}{-2n}\right) \\
&= \left(\frac{n-1+\sqrt{n^2+1}}{2n}, \frac{-2+2\sqrt{n^2+1}}{2n}\right).
\end{aligned}$$

For a_{4p+2}, we have $\frac{1-\alpha_{4p+2}}{\beta_{4p+2}} = \frac{n+1-\sqrt{n^2+1}}{-2+2\sqrt{n^2+1}}$.
Since $n = 4p+1 \geq 1$, we have:

$$\begin{aligned}
\frac{3}{4} &< n \\
6n &< 8n^2 \\
n^2+6n+9 &< 9n^2+9 \\
n+3 &< 3\sqrt{n^2+1} \\
n+1-\sqrt{n^2+1} &< -2+2\sqrt{n^2+1} \\
\frac{1-\alpha_{4p+2}}{\beta_{4p+2}} = \frac{n+1-\sqrt{n^2+1}}{-2+2\sqrt{n^2+1}} &< 1.
\end{aligned}$$

Also, we know that $n+1 > \sqrt{n^2+1}$, so:

$$\begin{aligned}
n+1-\sqrt{n^2+1} &> 0 \\
\frac{1-\alpha_{4p+2}}{\beta_{4p+2}} = \frac{n+1-\sqrt{n^2+1}}{-2+2\sqrt{n^2+1}} &> 0.
\end{aligned}$$

Thus $a_{4p+2} = 0$.
Therefore,

$$\begin{aligned}
(\alpha_{4p+3}, \beta_{4p+3}) &= \left(\frac{\beta_{4p+2}}{\alpha_{4p+2}}, \frac{1-\alpha_{4p+2}}{\alpha_{4p+2}}\right) \\
&= \left(\frac{-2+2\sqrt{n^2+1}}{n-1+\sqrt{n^2+1}}, \frac{n+1-\sqrt{n^2+1}}{n-1+\sqrt{n^2+1}}\right) \\
&= \left(\frac{-2n^2-2n+2n\sqrt{n^2+1}}{-2n}, \frac{2n^2-2n\sqrt{n^2+1}}{-2n}\right) \\
&= (n+1-\sqrt{n^2+1}, -n+\sqrt{n^2+1}).
\end{aligned}$$

Then

$$\frac{1-\alpha_{4p+3}}{\beta_{4p+3}} = \frac{-n+\sqrt{n^2+1}}{-n+\sqrt{n^2+1}} = 1.$$

So $a_{4p+3} = 1$, and the triangle sequence ends.

Case 2: $n = 4p+2$

We will show that the sequence ends with $a_{4p+1} = 1$.

If $n = 4p+2$, then $(\alpha_{4p+1}, \beta_{4p+1}) = (\frac{1-n+\sqrt{n^2+1}}{2}, \frac{1+n-\sqrt{n^2+1}}{2})$.

So:

$$\frac{1-\alpha_{4p+1}}{\beta_{4p+1}} = \frac{1+n-\sqrt{n^2+1}}{1+n-\sqrt{n^2+1}} = 1.$$

Therefore, $a_{4p+1} = 1$, and the sequence ends.

Case 3: $n = 4p+3$

We will show that the sequence ends with the terms $a_{4p+1} = 1, a_{4p+2} = 3$.

If $n = 4p+3$, then $(\alpha_{4p+1}, \beta_{4p+1}) = (\frac{1-n+\sqrt{n^2+1}}{2}, \frac{1+2(n-\sqrt{n^2+1})}{2})$.

So $\frac{1-\alpha_{4p+1}}{\beta_{4p+1}} = \frac{1+n-\sqrt{n^2+1}}{1+2n-2\sqrt{n^2+1}}$, and:

$$\begin{aligned}
\sqrt{n^2+1} &> n \\
1+n-\sqrt{n^2+1} &> 1+2n-2\sqrt{n^2+1} \\
\frac{1-\alpha_{4p+1}}{\beta_{4p+1}} = &> 1.
\end{aligned}$$

Also, $n \geq 3$, so $8 < 6n$, and:

$$\begin{aligned}
9n^2+9 &< 9n^2+6n+1 \\
3\sqrt{n^2+1} &< 3n+1 \\
1+n-\sqrt{n^2+1} &< 2+4n-4\sqrt{n^2+1} \\
\frac{1-\alpha_{4p+1}}{\beta_{4p+1}} &< 2.
\end{aligned}$$

So $a_{4p+1} = 1$.

Then:

$$\begin{aligned}
(\alpha_{4p+2}, \beta_{4p+2}) &= \left(\frac{1+2n-2\sqrt{n^2+1}}{1-n+\sqrt{n^2+1}}, \frac{-n+\sqrt{n^2+1}}{1-n+\sqrt{n^2+1}}\right) \\
&= \left(\frac{-3-n+3\sqrt{n^2+1}}{2n}, \frac{n+1-\sqrt{n^2+1}}{2n}\right)
\end{aligned}$$

But then we have:

$$\frac{1-\alpha_{4p+2}}{\beta_{4p+2}} = \frac{3+3n-3\sqrt{n^2+1}}{1+n-\sqrt{n^2+1}} = 3.$$

So $a_{4p+2} = 3$, and the sequence ends.

Case 4: $n = 4p + 4$:
We will show that the sequence ends with the terms $a_{4p+1} = 1, a_{4p+2} = 2$.
If $n = 4p + 4$, then $(\alpha_{4p+1}, \beta_{4p+1}) = (\frac{1-n+\sqrt{n^2+1}}{2}, \frac{1+3(n-\sqrt{n^2+1})}{2})$.
So for $a_{4p+1} = \left\lfloor \frac{1-\alpha_{4p+1}}{\beta_{4p+1}} \right\rfloor$, we have $\frac{1-\alpha_{4p+1}}{\beta_{4p+1}} = \frac{1+n-\sqrt{n^2+1}}{1+3n-3\sqrt{n^2+1}}$. We know $\sqrt{n^2+1} > n$, so we have:

$$\begin{aligned}
2\sqrt{n^2+1} &> 2n \\
1 + n - \sqrt{n^2+1} &> 1 + 3n - 3\sqrt{n^2+1} \\
\frac{1-\alpha_{4p+1}}{\beta_{4p+1}} &> 1
\end{aligned}$$

And $n \geq 4$, so:

$$\begin{aligned}
24 &< 10n \\
25n^2 + 25 &< 25n^2 + 10n + 1 \\
5\sqrt{n^2+1} &< 5n + 1 \\
1 + n - \sqrt{n^2+1} &< 2 + 6n - 6\sqrt{n^2+1} \\
\frac{1-\alpha_{4p+1}}{\beta_{4p+1}} &< 2
\end{aligned}$$

So $a_{4p+1} = 1$, and

$$\begin{aligned}
(\alpha_{4p+2}, \beta_{4p+2}) &= \left(\frac{1+3n-3\sqrt{n^2+1}}{1-n+\sqrt{n^2+1}}, \frac{-2n+2\sqrt{n^2+1}}{1-n+\sqrt{n^2+1}}\right) \\
&= \left(\frac{4+2n-4\sqrt{n^2+1}}{-2n}, \frac{-2n-2+2\sqrt{n^2+1}}{-2n}\right) \\
&= \left(\frac{-2-n+2\sqrt{n^2+1}}{n}, \frac{n+1-\sqrt{n^2+1}}{n}\right)
\end{aligned}$$

Then

$$\frac{1-\alpha_{4p+2}}{\beta_{4p+2}} = \frac{2+2n-2\sqrt{n^2+1}}{1+n-\sqrt{n^2+1}} = 2.$$

So $a_{4p+2} = 2$, and the sequence ends.

Thus the terms of the triangle sequence for $(\alpha, \beta) = (\frac{\sqrt{n^2+1}-n+1}{2}, \frac{n^2-n+1+(1-n)\sqrt{n^2+1}}{2})$ are as follows:

For $n \geq 4p + 5$:
$a_{4p+1} = 1 \qquad a_{4p+2} = 1 \qquad a_{4p+3} = 4p + 1 \qquad a_{4p+4} = n - 4p - 5$

For $n < 4p+5, n = 1 (\bmod 4)$:
$a_{4p+1} = 0 \qquad a_{4p+2} = 0 \qquad a_{4p+3} = 1 \qquad$ end

For $n < 4p+5, n = 2 (\bmod 4)$:
$a_{4p+1} = 1 \qquad$ end

For $n < 4p+5, n = 3 (\bmod 4)$:
$a_{4p+1} = 1 \qquad a_{4p+2} = 2 \qquad$ end

For $n < 4p+5, n = 0 (\bmod 4)$:
$a_{4p+1} = 1 \qquad a_{4p+2} = 3 \qquad$ end

$\square$

References

[1] Thomas Garrity, On Periodic Sequences for Algebraic Numbers. 1999.

[2] T. Cheslack-Postava, A. Diesl, M. Lepinski, A. Schuyler, Some Results Concerning Uniqueness of Triangle Sequences. 1999.