On periodic sequences for algebraic numbers

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Abstract

For each positive integer $n \geq 2$, a new approach to expressing real numbers as sequences of nonnegative integers is given. The $n = 2$ case is equivalent to the standard continued fraction algorithm. For $n = 3$, it reduces to a new iteration of the triangle. Cubic irrationals that are roots of $x^3 + kx^2 + x - 1$ are shown to be precisely those numbers with purely periodic expansions of period length one. For general positive integers n, it reduces to a new iteration of an n dimensional simplex.

1 Introduction

In 1848 Hermite [5] posed to Jacobi the problem of generalizing continued fractions so that periodic expansions of a number reflect its algebraic properties. We state this as:

The Hermite Problem: *Find methods for writing numbers that reflect special algebraic properties.*

In attempting to answer this question, Jacobi developed a special case of what is now called the Jacobi-Perron algorithm. Bernstein [1] wrote a good survey of this algorithm; Schweiger [9] covered its ergodic properties. Brentjes' book [2] is a good source for its many variations. Using quite different methods, Minkowski [7] developed a quite different approach to the Hermite's problem. For another attempt, see the work of Ferguson and Forcade [4].

In this paper we give another approach, which will also be a generalization of continued fractions. To each n-tuple of real numbers $(\alpha_1, \ldots, \alpha_n)$, with $1 \geq \alpha_1 \geq \ldots \geq \alpha_n$, we will associate a sequence of nonnegative integers. For reasons that will become apparent later, we will call this sequence the *triangle sequence* (or *simplex sequence*) for the n-tuple. The hope is that the periodicity of this sequence will provide insight into whether or not the α_k are algebraic of degree at most n. We will show that this is the case for when $n = 3$.

In the next section we quickly review some well-known facts about continued fractions. We then concentrate on the cubic case, for ease of exposition. The proofs go over easily to the general case, which we will discuss in section nine. In section three we define, given a pair $(\alpha, \beta) \in \{(x, y) : 1 \geq x \geq y \geq 0\}$, the triangle iteration and the triangle sequence. Section four will recast the triangle sequence via matrices. This will allow us to interpret the triangle sequence as a method for producing integer lattice points that approach the plane $x + \alpha y + \beta z = 0$. Section five will show that nonterminating triangle sequences uniquely determine the pair (α, β).

Section six discusses how every possible triangle sequence corresponds to a pair $(\alpha, \beta) \in \{(x, y) : 1 \geq x \geq y \geq 0\}$. Section seven turns to classifying those pairs with purely periodic sequences. Section eight concerns itself with periodicity in general. Section nine deals with the general case n.

At http://www.williams.edu/Mathematics/tgarrity/triangle.html, there is a web page that give many examples of triangle sequences and provides software packages running on Mathematica for making actual computations.

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2 Continued Fractions

Given a real number α, recall that its continued fraction expansion is:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}},$$

where $a_0 = \{\alpha\}$ = greatest integer part of α,

$$a_1 = \{\frac{1}{\alpha - a_0}\} \text{ and } b_1 = \frac{1}{\alpha - a_0} - a_1.$$

Inductively, define

$$a_k = \{\frac{1}{b_{k-1}}\} \text{ and } b_k = \frac{1}{b_{k-1}} - a_k.$$

A number's continued fraction expansion can be captured by examining iterations of the Gauss map $G : I \to I$, with I denoting the unit interval $(0, 1]$, defined by

$$G(x) = \frac{1}{x} - \{\frac{1}{x}\}.$$

If we partition the unit interval into a disjoint union of subintervals:

$$I_k = \{x \in I : \frac{1}{k+1} < x \leq \frac{1}{k}\},$$

then the nonnegative integers a_k in the continued fraction expansion of α can be interpreted as keeping track of which subinterval the number α maps into under the kth iterate of G. Namely, $G^k(\alpha) \in I_{a_k}$.

3 The Triangle Iteration

In this section we define an iteration T on the triangle

$$\triangle = \{(x, y) : 1 \geq x \geq y > 0\}.$$

Partition this triangle into disjoint triangles

$$\triangle_k = \{(x, y) \in \triangle : 1 - x - ky \geq 0 > 1 - x - (k+1)y\},$$

where k can be any nonnegative integer. Note that its vertices are $(1, 0)$, $(\frac{1}{k+1}, \frac{1}{k+1})$ and $(\frac{1}{k+2}, \frac{1}{k+2})$.

Define the *triangle map* $T : \triangle \to \triangle \cup \{(x, 0) : 0 \leq x \leq 1\}$ by setting

$$T(\alpha, \beta) = (\frac{\beta}{\alpha}, \frac{1 - \alpha - k\beta}{\alpha}),$$

if the pair $(\alpha, \beta) \in \triangle_k$. Frequently we will abuse notation by denoting $\triangle \cup \{(x, 0) : 0 \le x \le 1\}$ by $\triangle$.

We want to associate a sequence of nonnegative integers to the iterates of the map T. Basically, if $T^k(\alpha, \beta) \in \triangle_{a_k}$, we will associate to (α, β) the sequence $(a_1, \ldots)$.

Recursively define a sequence of decreasing positive reals and a sequence of nonnegative integers as follows: Set $d_{-2} = 1, d_{-1} = \alpha, d_0 = \beta$. Assuming that we have $d_{k-3} > d_{k-2} > d_{k-1} > 0$, define a_k to be a nonnegative integer such that

$$d_{k-3} - d_{k-2} - a_k d_{k-1} \ge 0$$

but

$$d_{k-3} - d_{k-2} - (a_k + 1) d_{k-1} < 0.$$

Then set

$$d_k = d_{k-3} - d_{k-2} - a_k d_{k-1}.$$

If at any stage $d_k = 0$, stop.

Definition 1 *The triangle sequence of the pair* (α, β) *is the sequence* $(a_1, \ldots)$.

We will say that the triangle sequence *terminates* if at any stage $d_k = 0$. In these cases, the triangle sequence will be finite.

Note that

$$T(\frac{d_{k-1}}{d_{k-2}}, \frac{d_k}{d_{k-2}}) = (\frac{d_k}{d_{k-1}}, \frac{d_{k+1}}{d_{k-1}}).$$

Also note that by comparing this to the first part of chapter seven in [10], we see that this is indeed a generalization of continued fractions.

4 The Triangle Iteration via Matrices and Integer Lattice Points

Let $(a_1, \ldots)$ be a triangle sequence associated to the pair (α, β). Set

$$P_k = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -a_k \end{pmatrix}.$$

Note that $\det P_k = 1$. Set $M_k = P_1 \cdot P_2 \cdots P_k$. This allows us to translate the fact that

$$T(\frac{d_{k-1}}{d_{k-2}}, \frac{d_k}{d_{k-2}}) = (\frac{d_k}{d_{k-1}}, \frac{d_{k+1}}{d_{k-1}})$$

into the language of matrices via the following proposition (whose proof is straightforward):

Proposition 2 *Given the pair (α, β), we have*

$$(d_{k-2}, d_{k-1}, d_k) = (1, \alpha, \beta) M_k.$$

Write

$$M_k = \begin{pmatrix} p_{k-2} & p_{k-1} & p_k \\ q_{k-2} & q_{k-1} & q_k \\ r_{k-2} & r_{k-1} & r_k \end{pmatrix}.$$

Then a calculation leads to:

Proposition 3 *For all k, we have*

$$p_k = p_{k-3} - p_{k-2} - a_k p_{k-1},$$

$$q_k = q_{k-3} - q_{k-2} - a_k q_{k-1},$$

and

$$r_k = r_{k-3} - r_{k-2} - a_k r_{k-1}.$$

Set

$$C_k = \begin{pmatrix} p_k \\ q_k \\ r_k \end{pmatrix}.$$

Note that C_k can be viewed as a vector in the integer lattice. Then the numbers d_k are seen to be a measure of the distance from the plane $x + \alpha y + \beta z = 0$ to the lattice point C_k, since $d_k = (1, \alpha, \beta)C_k$. Observing that

$$C_k = C_{k-3} - C_{k-2} - a_k C_{k-1},$$

we see thus that the triangle sequence encodes information of how to get a sequence of lattice points to approach the plane $x + \alpha y + \beta z = 0$, in direct analogue to continued fractions [10]. Unlike the continued fraction case, though, these lattice points need not be the best such approximations.

5 Arbitrary triangle sequences

Theorem 4 *Let $(k_1, k_2, \ldots)$ be any infinite sequence of nonnegative integers with infinitely many of the k_i not zero. Then there is a pair (α, β) in $\triangle$ that has this sequence as its triangle sequence.*

Proof: Suppose that we have an infinite triangle sequence $(k_1, k_2, \ldots)$. By a straighforward calculation,we see that a line with equations $y = mx + b$ will map to the line

$$(1-kb)u-(1-kb)m = bv + bkm + b,$$

where $T(x,y)=(u,v)$.

The map T, restricted to the triangle $\triangle_k$, will send the vertices of $\triangle_k$ to the vertices of $\triangle$, with $T(1,0)=(0,0)$, $T(\frac{1}{k+1},\frac{1}{k+1})=(1,0)$ and $T(\frac{1}{k+2},\frac{1}{k+2})=(1,1)$. Restricted to $\triangle_k$, the map T is thus one-to-one and onto $\triangle$.

But this gives us our theorem, as each $\triangle_k$ can be split into its own (smaller) triangles, one for each nonnegative integer, and hence each of these smaller triangles can be split into even smaller triangles, etc. Hence to each nonterminating triangle sequence there corresponds a pair (α,β). **QED**

6 Recovering points from the triangle sequence

The question of when a triangle sequence determines a unique pair (α,β) is subtle. If the sequence terminates, then the pair (α,β) is not unique. Even if the triangle sequence does not terminate, we do not necessarily have uniqueness, as discussed in [3]. But we do have

Theorem 5 *If an integer k occurs infinitely often in a nonterminating sequence $(k_1,k_2,\ldots)$ of nonnegative integers, then there is a unique pair (α,β) in $\triangle$ that has this sequence as its triangle sequence.*

The proof is contained in [3] and is not easy.

If the triangle sequence uniquely determines a pair (α, β), then we can recover (α, β) as follows. By construction, the numbers d_k approach zero. Consider the plane

$$x + \alpha y + \beta z = 0,$$

whose normal vector is $(1, \alpha, \beta)$. As seen in the last section, the columns of the matrices M_k can be interpreted as vectors that are approaching this plane. This will allow us to prove:

Theorem 6 *If a triangle sequence uniquely determines the pair (α, β), then*

$$\alpha = \lim_{k \to \infty} \frac{p_k r_{k-1} - p_{k-1} r_k}{q_{k-1} r_k - q_k r_{k-1}}$$

and

$$\beta = \lim_{k \to \infty} \frac{p_{k-1} q_k - p_k q_{k-1}}{q_{k-1} r_k - q_k r_{k-1}}.$$

The proof is also in [3]. The quick, but incorrect, argument is that the vectors $(p_{k-1}, q_{k-1}, r_{k-1})$ and (p_k, q_k, r_k) are columns in the matrix M_k, each of which approaches being in the plane $x + \alpha y + \beta z = 0$. Thus the limit as k approaches infinity of the cross product of these two vectors must point in the normal direction $(1, \alpha, \beta)$. But this is the above limits.

7 Purely periodic triangle sequences of period length one

Theorem 7 *Let $0 < \beta \leq \alpha < 1$ be a pair of numbers whose triangle sequence is $(k, k, k, \ldots)$. Then $\beta = \alpha^2$ and α is a root of the cubic equation*

$$x^3 + kx^2 + x - 1 = 0.$$

Further if α is the real root of this cubic that is between zero and one, then (α, α^2) has purely periodic triangle sequence $(k, k, k, \ldots)$.

Proof: We need $T(\alpha, \beta) = (\alpha, \beta)$. Since $T(\alpha, \beta) = (\frac{\beta}{\alpha}, \frac{1-\alpha-k\beta}{\alpha})$, we need

$$\alpha = \frac{\beta}{\alpha}$$

and

$$\beta = \frac{1 - \alpha - k\beta}{\alpha}.$$

From the first equation we get $\beta = \alpha^2$. Plugging in α^2 for β in the second equation and clearing denominators leads to

$$\alpha^3 + k\alpha^2 + \alpha - 1 = 0$$

and the first part of the theorem.

Now for the converse. Since the polynomial $x^3 + kx^2 + x - 1$ is -1 at $x = 0$ and is positive at $x = 1$, there is root α between zero and one. We must show that (α, α^2) is in $\triangle_k$ and that $T(\alpha, \alpha^2) = (\alpha, \alpha^2)$. We know that

$$\alpha^3 = 1 - \alpha - k\alpha^2.$$

Since $\alpha^3 > 0$, we have $1 - \alpha - k\alpha^2 > 0$. Now

$$1 - \alpha - (k+1)\alpha^2 = \alpha^3 - \alpha^2 < 0,$$

which shows that $(\alpha, \alpha^2) \in \triangle_k$.

Finally,

$$\begin{aligned} T(\alpha, \alpha^2) &= (\frac{\alpha^2}{\alpha}, \frac{1 - \alpha - k\alpha^2}{\alpha}) \\ &= (\alpha, \alpha^2). \end{aligned}$$

QED

Similar formulas for purely periodic sequences with period length two, three, etc., can be computed, but they quickly become computationally messy.

8 Terminating and Periodic Triangle Sequences

We first want to show that if (α, β) is a pair of rational numbers, then the corresponding triangle sequence must terminate, meaning that eventually all of the k_n will be zero.

Theorem 8 *Let (α, β) be a pair of rational numbers in $\triangle$. Then the corresponding triangle sequence terminates.*

Proof: In constructing the triangle sequence, we are just concerned with the ratios of the triple $(1, \alpha, \beta)$. By clearing denominators, we can replace this triple by a triple of positive integers (p, q, r), with $p \geq q \geq r$. Then we have $d_{-2} = p, d_{-1} = q, d_0 = r$. Then the sequence of d_k will be a sequence of positive decreasing integers. Thus for some k we must have $d_k = 0$, forcing the triangle sequence to terminate.

QED

Now to see what happens when the triangle sequence is eventually periodic.

Theorem 9 *Let (α, β) be a pair of real numbers in $\triangle$ whose triangle sequence is eventually periodic. Then α and β have degree at most three, with $\alpha \in \mathbf{Q}[\beta]$ or $\beta \in \mathbf{Q}[\alpha]$.*

Proof: If both α and β are rational, then by the above theorem the triangle sequence terminates. Thus we assume that not both α and and β are rational. Since the triangle sequence is periodic, there will be an integer appearing infinitely often in this sequence, which means that the sequence will uniquely determine a pair (α, β).

If the triangle sequence is periodic, there must be an n and m so that

$$(\frac{d_{n-2}}{d_n}, \frac{d_{n-1}}{d_n}) = (\frac{d_{m-2}}{d_m}, \frac{d_{m-1}}{d_m}).$$

Thus there exists a number λ with

$$(d_{n-2}, d_{n-1}, d_n) = \lambda(d_{m-2}, d_{m-1}, d_m).$$

Using matrices we have:

$$(1, \alpha, \beta)M_n = \lambda(1, \alpha, \beta)M_m$$

and thus

$$(1, \alpha, \beta) M_n M_m^{-1} = \lambda (1, \alpha, \beta).$$

Since M_n and M_m have integer coefficients, the matrix $M_n M_m^{-1}$ will have rational coefficients. Since the d_k are decreasing, we must have $|\lambda| \neq 1$. Since both M_n and M_m have determinant one, we have that $M_n M_m^{-1}$ cannot be a multiple of the identity matrix.

Set

$$M_n M_m^{-1} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}.$$

Then

$$q_{11} + q_{21}\alpha + q_{31}\beta = \lambda$$

$$q_{12} + q_{22}\alpha + q_{32}\beta = \lambda\alpha$$

and

$$q_{13} + q_{23}\alpha + q_{33}\beta = \lambda\beta.$$

We can eliminate the unknown λ from the first and second equations and then from the first and third equations, leaving two equations with unknowns α and β. Using these two equations we can eliminate one of the remaining variables, leaving the last as the solution to polynomial with rational coefficients. If this polynomial is the zero polynomial, then it can be seen that this will force $M_n M_m^{-1}$ to be a multiple of the identity, which is not possible. Finally, it can be checked that this polynomial is a cubic.

QED

9 The higher degree case

Almost all of this goes over in higher dimensions. We just replace our triangle by a dimension n simplex. The notation, though, is more cumbersome.

Set

$$\triangle = \{(x_1, \ldots, x_n) : 1 \geq x_1 \geq \ldots \geq x_n > 0\}.$$

As we did before, we will frequently also call $\triangle = \{(x_1, \ldots, x_n) : 1 \geq x_1 \geq \ldots \geq x_n \geq 0\}$. Set

$$\triangle_k = \{(x_1, \ldots, x_n) \in \triangle : 1-x_1-\ldots-x_{n-1}-kx_n \geq 0 > 1-x_1-\ldots-x_{n-1}-(k+1)x_n\},$$

where k can be any nonnegative integer. These are the direct analogue of the triangles $\triangle_k$ in the first part of this paper. Unlike the earlier case, these $\triangle_k$, while disjoint, do not partition the simplex $\triangle$. To partition $\triangle$, we need more simplices. Set

$$\triangle' = \{(x_1, \ldots, x_n) \in \triangle : 0 > 1 - x_1 - \ldots - x_n\}.$$

Then set

$$\triangle_{ij} = \{(x_1, \ldots, x_n) \in \triangle' : x_j \geq 1 - x_1 - \ldots - x_i \geq x_{j+1}\},$$

where $1 \leq i \leq n-2$ and $i < j \leq n$. Also, we use the convention that x_{n+1} is identically zero.

Lemma 10 *The $\triangle_k$ and $\triangle_{ij}$ form a simplicial decomposition of the simplex $\triangle$.*

Proof: It can be directly checked that the $\triangle_k$ and $\triangle_{ij}$ do form a disjoint partition of $\triangle$. We need to show that the $\triangle_k$ and $\triangle_{ij}$ are simplices. Thus we want to show that each of these polygons have exactly $n+1$ vertices. Label the $n+1$ vertices of the simplex $\triangle$ by $v_0 = (0,\ldots,0), v_1 = (1,0,\ldots,0), v_2 = (1,1,0,\ldots,0),\ldots, v_n = (1,\ldots,1)$. We label each of the $\frac{n(n+1)}{2}$ edges of the simplex by v_iv_j if the endpoints of the edge are the vertices v_i and v_j. Consider the set $\triangle_k$. The hyperplanes

$$x_1 + \ldots + x_{n-1} + kx_n = 1$$

and

$$x_1 + \ldots + x_{n-1} + (k+1)x_n = 1$$

form two of the faces. These hyperplanes intersect each edge v_0v_l, with $l < n$, in the same point $(\frac{1}{l},\ldots,\frac{1}{l},0,\ldots,0)$, where this n-tuple has its first l terms $\frac{1}{l}$ and the rest zero. The hyperplanes intersect the edge v_0v_n in two distinct points: $x_1 + \ldots + x_{n-1} + kx_n = 1$ intersects at the point $(\frac{1}{n+k-1},\ldots,\frac{1}{n+k-1})$ while $x_1 + \ldots + x_{n-1} + (k+1)x_n = 1$ intersects in the point $(\frac{1}{n+k},\ldots,\frac{1}{n+k})$. Since both hyperplanes contain the vertex v_1, both intersect all of the edges v_1v_l exactly at v_1. Both hyperplanes will miss all of the other edges v_iv_j, with $i,j \geq 2$, since for every point on all of these edges, $x_1 = 1$, $x_2 = 1$ and $x_l \geq 0$, forcing the intersections to be empty. But now we just have to count and see that the number of vertices is indeed $n+1$. Thus $\triangle_k$ is a simplex.

The argument is similar for $\triangle_{ij}$. Here we look at the hyperplanes

$$x_1 + \ldots + x_i + x_j = 1$$

and

$$x_1 + \ldots + x_{n-1} + x_{j+1} = 1.$$

Both will intersect each of the edges $v_0 v_l$, for $l \neq j$, in the same point, will intersect the edges $v_1 v_l$ exactly at v_1 and will miss the edges $v_i v_j$, with $i, j \geq 2$. They will intersect the edge $v_0 v_j$ at distinct points. Then $\triangle_{ij}$ has $n+1$ distinct vertices and is thus a simplex.

QED

Define the *n-triangle map* $T : \triangle \to \triangle$ by setting

$$T(\alpha_1, \ldots, \alpha_n) = (\frac{\alpha_2}{\alpha_1}, \ldots, \frac{\alpha_{n-1}}{\alpha_1}, \frac{1 - \alpha_1 \ldots - \alpha_{n-1} - k\alpha_n}{\alpha_1}),$$

if $(\alpha_1, \ldots, \alpha_n) \in \triangle_k$ and by

$$T(\alpha_1, \ldots, \alpha_n) = (\frac{\alpha_2}{\alpha_1}, \ldots, \frac{\alpha_j}{\alpha_1}, \frac{1 - \alpha_1 \ldots - \alpha_i}{\alpha_1}, \frac{\alpha_{j+1}}{\alpha_1}, \ldots, \frac{\alpha_n}{\alpha_1}),$$

if $(\alpha_1, \ldots, \alpha_n) \in \triangle_{ij}$.

By direct calculation, we see that $T(\alpha_1, \ldots, \alpha_n) \in \triangle$. Further, each of the restriction maps $T : \triangle_k \to \triangle$ and $T : \triangle_{ij} \to \triangle$ are one-to-one and onto, since the vertices of $\triangle_k$ and $\triangle_{ij}$ map to the vertices of $\triangle$ and since lines map to lines.

We want to associate to each $(\alpha_1, \ldots, \alpha_n)$ in $\triangle$ an infinite sequence $(a_0, a_1, \ldots)$, where each a_k is either a non-negative integer or a symbol (ij), where $1 \leq i \leq n-2$ and $i < j \leq n$. If $T^k(\alpha_1, \ldots, \alpha_n) \in \triangle_l$, set $a_k = l$ and if $T^k(\alpha_1, \ldots, \alpha_n) \in \triangle_{ij}$, set $a_k = (ij)$. Finally, if the nth term for $T^k(\alpha_1, \ldots, \alpha_n)$ is zero, stop.

Definition 11 *The n-triangle sequence of $(\alpha_1, \ldots, \alpha_n)$ is the sequence $(a_1, \ldots)$.*

We can also recursively define the triangle sequence as follows. We want to define a sequence of $(n+1)$-tuples of nonincreasing positive numbers. We will denote this sequence by $d_1(k), \ldots, d_{n+1}(k)$, for $k \geq 0$. Start with

$$d_1(0) = 1, d_2(0) = \alpha_1, \ldots, d_{n+1}(0) = \alpha_n.$$

Assume we have $d_1(k-1), \ldots, d_{n+1}(k-1)$. Define the symbol a_k as follows. If there is a nonnegative integer l such that

$$d_1(k-1) - d_2(k-1) - \ldots - d_n(k-1) - l d_{n+1}(k-1) \geq 0$$

but

$$d_1(k-1) - d_2(k-1) - \ldots - d_n(k-1) - (l+1) d_{n+1}(k-1) < 0,$$

set $a_k = l$ and define

$$d_1(k) = d_2(k-1), \ldots, d_n(k) = d_{n+1}(k-1)$$

and

$$d_{n+1}(k) = d_1(k-1) - d_2(k-1) - \ldots - d_n(k-1) - l d_{n+1}(k-1).$$

If no such integer exists, then there is a pair (ij), with $1 \leq i \leq n-1$ and $i < j \leq n+1$ such that

$$d_j(k-1) \geq d_1(k-1) - d_2(k-1) - \ldots - d_i(k-1) > d_{j+1}(k-1).$$

In this case, define $a_k = (ij)$ and set

$$d_1(k) = d_2(k-1), \ldots, d_{j-1}(k) = d_j(k-1),$$

$$d_j(k) = d_1(k-1) - d_2(k-1) - \ldots - d_i(k-1)$$

and

$$d_{j+1}(k) = d_{j+1}(k-1), \ldots, d_{n+1}(k) = d_{n+1}(k-1).$$

Now for the matrix version. Let $(a_1, \ldots)$ be an n-triangle sequence for $(\alpha_1, \ldots, \alpha_n)$. If a_k is a nonnegative integer, let P_k be the $(n+1) \times (n+1)$

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & -1 \\ & \vdots & & & \\ 0 & \ldots & 1 & 0 & -1 \\ 0 & \ldots & 0 & 1 & -a_k \end{pmatrix}.$$

If a_k is the pair (ij), let P_k be the $(n+1) \times (n+1)$ matrix defined by:

$$(x_1, \ldots, x_{n+1})P_k = (x_2, \ldots x_j, x_1 - x_2 - \ldots - x_i, x_{j+1}, \ldots, x_{n+1}).$$

Then set $M_k = P_1 \cdot P_2 \cdots P_k$. Note that $\det M_k = \pm 1$. We have

$$(d_1(k), \ldots d_{n+1}(k)) = (1, \alpha_1, \ldots, \alpha_n)M_k.$$

Set $M_k = (C_1(k), \ldots, C_{n+1}(k))$, where each $C_m(k)$ is a column vector of the matrix. Then, if $a_k = l$,

$$C_m(k) = C_{m+1}(k-1)$$

for $m \leq n$ and

$$C_{n+1}(k) = C_1(k-1) - C_2(k-1) - \ldots - C_n(k) - lC_{n+1}(k-1).$$

If $a_k = (ij)$, then, for $1 \leq m \leq j+1$,

$$C_{m-1}(k) = C_m(k-1),$$

$$C_{j+1}(k) = C_1(k-1) - C_2(k-1) - \ldots - C_{i+1}(k),$$

and for $j+1 \leq m \leq n+1$,

$$C_m(k) = C_m(k-1).$$

Each $C_k(m)$ can be viewed as an element of the integer lattice Z^{n+1}. Then we have a method for producing elements of the integer lattice that approach the hyperplane

$$x_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n = 0.$$

It is still unknown how to determine when an n-triangle sequence will uniquely determine an n-tuple $(\alpha_1, \ldots, \alpha_n) \in \triangle$. If we have uniqueness, we strongly suspect that

$$\alpha_j = \lim_{k \to \infty} (-1)^j \frac{M_k(j1)}{M_k(11)},$$

where $M_k(ij)$ denotes the determinant of the $n \times n$ minor of M_k obtained by deleting the ith row and jth column. The moral, but incorrect argument, is the following. First, since $\det M_k = \pm 1$, its column vectors are linearly independent. But also, the column vectors are approaching the hyperplane whose normal vector is $(1, \alpha_1, \ldots, \alpha_n)$. Then via standard arguments, the wedge product $C_2(k) \wedge \ldots \wedge C_{n+1}(k)$ corresponds under duality to a vector perpendicular to $C_2(k), \ldots, C_{n+1}(k)$ and by normalizing, will approach the vector $(1, \alpha_1, \ldots, \alpha_n)$

With reasonable conditions about uniqueness, we should have

Conjecture 12 *Let $0 \leq \alpha_n \leq \ldots \leq \alpha_1 < 1$ be an n-tuple of numbers whose triangle sequence is $(k, k, k, \ldots)$. Then $\alpha_j = \alpha_1^j$ and α_1 is a root of the algebraic equation*

$$x^{n+1} + kx^n + x^{n-1} + \ldots + x - 1 = 0.$$

Further if α is the real root of this equation that is between zero and one, then $(\alpha, \alpha^2, \ldots, \alpha^n)$ has purely periodic simplex sequence $(k, k, k, \ldots)$.

A similar result holds if the triangle sequence is purely periodic of period length one of the form $(ij, ij, ij, \ldots)$.

We should also have

Conjecture 13 *Let $(\alpha_1, \ldots, \alpha_n)$ be an n-tuple real numbers in $\triangle$ whose triangle sequence is eventually periodic. Then each α_j is algebraic of degree at* *most n. Finally, as a vector space over $\mathbf{Q}$, the dimension of $\mathbf{Q}[\alpha_1, \ldots, \alpha_n]$ is at most n.*

Finally a comment about notation. In the abstract it is claimed that that we will express each n-tuple will be associated to a sequence of nonnegative integers but in this section we look at sequences not just of nonnegative integers but also of terms of the form (ij) with $1 \leq i \leq n-2$ and $i < j \leq n$. But there are only a finite number ($n(n-2) + \frac{(n-2)(n-1)}{2}$) of these extra symbols. We could, if desired, order these symbols by nonnegative numbers $0, 1, \ldots, n(n-2) + \frac{(n-2)(n-1)}{2}$ and then shift the original nonnegative integers by this amount. This will force the sequence to be one of nonnegative integers, but the notation is clearly worse than the one chosen.

References

[1] L.Bernstein, The Jacobi-Perron algorithm-its theory and application, Lecture Notes in Mathematics,**207**,*Springer-Verlag*, Berlin, 1971.

[2] A.J.Brentjes, Multi-dimensional Continued Fraction Algorithms, *Mathemaical Centre Tracts*, Amsterdam, 1981.

[3] T. Cheslack-Postava, A. Diesl, T. Garrity, M. Lepinski, A. Schuyler, Some Results Concerning Uniqueness of Triangle Sequences, in preparation (a preliminary draft can be found at http://www.williams.edu/Mathematics/tgarrity/triangle.html)

[4] H.R.P. Ferguson and R.W. Forcade, Generalization of the Euclidean algorithm for real numbers to all dimensions higher than two, *Bull. of the AMS* Vol 1, n. 6 (1979), pp 912-914.

[5] C. Hermite, 'Letter to C.D.J. Jacobi', *J.f.d.reine angew Math.*, **40**, 286, 1839.

[6] K. Mahler, Inequalities for Ideal Bases in Algebraic Number Fields, *Jour. of the Australian Math. Soc.* Vol. 4, 1964, pp425-448.

[7] H. Minkowski, Ein Kriterium für algebraishcen Zahlen, *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse*, pp. 64-88. (Also in Gesammelte Abhandlungen, Chelsea, pp.293-315).

[8] H. Minkowski, Über periodische Approximationen algebraischer Zahlen, *Acta Mathematica*, BD. 26, S 333-351. (Also in Gesammelte Abhandlungen, Chelsea, pp. 357-371).

[9] F. Schweiger, The metrical theory of Jacobi-Perron algorithm, Lecture Notes in Mathematics, **334**,*Springer-Verlag*, Berlin, 1973.

[10] H. Stark, An Introduction to Number Theory, *MIT Press*, Cambrdge, 1994.