Some Results Concerning Uniqueness of Triangle Sequences

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Abstract

In this paper, we will begin by reviewing the triangle iteration. We will then investigate the question of when a triangle sequence corresponds to a unique ordered pair (α, β). It will be shown that certain classes of triangle sequences, most importantly those that are bounded, do yield unique ordered pairs. As a corollary to our work on uniqueness, we will also show some clean relations dealing with the geometry of the partition triangles.

1 Introduction

This paper will start with an overview of the standard triangle iteration as outlined in work by Garrity [1]. We will begin with the geometrical interpretation of triangle sequences. We define an iteration T on the triangle

$$\triangle = \{(x, y) : 1 \geq x \geq y > 0\}.$$

This triangle is partitioned into an infinite set of disjoint subtriangles

$$\triangle_k = \{(x, y) \in \triangle : 1 - x - ky \geq 0 > 1 - x - (k+1)y\},$$

where k is any nonnegative integer.

Given $(\alpha, \beta) \in \triangle_k$, define the map $T : \triangle \to \triangle \cup \{(x, 0) : 0 \leq x \leq 1\}$ by

$$T(\alpha, \beta) = \left(\frac{\beta}{\alpha}, \frac{1 - \alpha - k\beta}{\alpha}\right).$$

The triangle sequence is recovered from this iteration by keeping track of the number of the triangle that the point is mapped into at each step. In other words, if $T^{k-1}(\alpha, \beta) \in \triangle_{a_k}$, then the point (α, β) will have the triangle sequence $(a_1, a_2, \dots)$.

We will recursively define a sequence of vectors as follows: Set $C_{-2} = (1, 0, 0)$, $C_{-1} = (0, 1, 0)$ and $C_0 = (0, 0, 1)$. Let the components of C_n be denoted by $C_n = (p_n, q_n, r_n)$. Then let

$$C_n = C_{n-3} - C_{n-2} - a_n C_{n-1}.$$

These C_n vectors can be thought of as integer vectors approximating the plane $x + \alpha y + \beta z = 0$. We thus refer to the C_n vectors as *approximation vectors*. We define positive numbers d_n in the following manner:

$$d_n = (1, \alpha, \beta) \cdot C_n.$$

These are interpreted as the distance from the vector C_n to the plane $x + \alpha y + \beta z = 0$. It is then easy to see that the following recursion relations hold:

$$\begin{aligned} d_n &= d_{n-3} - d_{n-2} - a_n d_{n-1} \\ p_n &= p_{n-3} - p_{n-2} - a_n p_{n-1} \\ q_n &= q_{n-3} - q_{n-2} - a_n q_{n-1} \\ r_n &= r_{n-3} - r_{n-2} - a_n r_{n-1} \end{aligned}$$

This paper will focus on the problem of discovering when a particular sequence $(a_1, a_2, \dots)$ corresponds to a unique pair (α, β), which has implications in determining if α and β are cubic irrationals. We consider only the case of infinite triangle sequences because it is trivial to show that all finite triangle sequences correspond to an infinite number of points, and are thus not unique. The proof of this assertion will appear in section 2, after appropriate notation has been introduced.

There are two main ways to frame the question of uniqueness. The first relies on vector algebra. We know that the vectors C_n are approaching the plane $x + \alpha y + \beta z = 0$. It is also easy to see that the C_n vectors are determined completely by the sequence $(a_1, a_2, \dots)$. Thus, if it can be shown that the cross product $C_n \times C_{n+1}$ of two successive approximation vectors approaches the vector $(1, \alpha, \beta)$, then the sequence is unique. In other words, if

$$\lim_{n \to \infty} \frac{C_n \times C_{n+1}}{||C_n \times C_{n+1}||} = (1, \alpha, \beta),$$

then the numbers α and β will be uniquely determined by the triangle sequence $(a_1, a_2, \dots)$.

The second approach to understanding the uniqueness problem relies on the geometry of the actual triangle iteration. We consider the original partitioning of the triangle into an infinite disjoint union of subtriangles:

$$\triangle = \bigcup_{k=0}^{\infty} \triangle_k$$

It is clear from the definition of triangle sequences that $\triangle_k$ consists of all points (α, β) whose triangle sequences have first term equal to k. In other words,

$$\triangle_k = \{(x, y) \in \triangle : a_1 = k\}.$$

We can further partition each $\triangle_k$ in a way similar to that in which we partitioned $\triangle$. Define $\triangle_{k,l}$ in the following way:

$$\triangle_{k,l} = \{(x,y) \in \triangle_k : T(x,y) \in \triangle_l\}.$$

Thus $\triangle_{k,l}$ is the set of all points in $\triangle_k$ that map to $\triangle_l$. We can think of this set in the following way:

$$\triangle_{k,l} = \{(x,y) \in \triangle : a_1 = k \text{ and } a_2 = l\}.$$

We can further partition the triangles in the same way. We define $\triangle_{a_1,a_2,\dots,a_n}$ to be the set of points (α,β) whose triangle sequences have $(a_1,a_2,\dots,a_n)$ as their first n terms. This gives us a method for investigating uniqueness. If we can say that the set of points with the infinite triangle sequence $(a_1,a_2,\dots)$ consists of only one point, (α,β), then we can say that (α,β) is the unique point with that triangle sequence. One way of investigating the set $\triangle_{a_1,a_2,\dots}$ is to look at the limit of the sets $\triangle_{a_1,a_2,\dots,a_n}$ as n approaches infinity.

Section 2 of this paper uses a geometric argument to show that there do, in fact, exist certain infinite triangle sequences which correspond to an infinite number of points (α,β). This will be demonstrated by showing that, in certain cases, the set $\triangle_{a_1,a_2,\dots}$ consists of an entire line segment.

Section 3 is devoted to a development of the vector approach to the uniqueness problem. We will define new notation for cross products of successive approximation vectors. We will then prove a new recursion relation and develop algebraic conditions that guarantee uniqueness.

Section 4 uses the algebraic conditions developed in section 3 in order to prove uniqueness of certain bounded triangle sequences. Specifically, we prove that an infinite triangle sequence $(a_1,a_2,\dots)$ corresponds to a unique ordered pair (α,β) for the case $a_i = A$ for all i and some positive integer A.

Section 5 explores the connections between the vector algebra and the geometry of the partitioning of the triangles. We show that there is a clean relation between the cross product vectors and the vertices of the partition triangles $\triangle_{a_1,a_2,\dots,a_n}$. We will also show that there is an interesting relationship between the vertices of $\triangle_{a_1,a_2,\dots,a_n}$ and the vertices of $\triangle_{a_n,a_{n-1},\dots,a_1}$. These results are interesting in their own right, but will be mainly used as technical lemmae in proving the main theorem of section 6.

Section 6 deals with the case of proving the uniqueness of a larger class of triangle sequences, including the entire bounded case. Specifically, we show that, for any infinite sequence in which the integer C appears infinitely many times, that sequence corresponds to a unique pair (α,β). We use both geometric and algebraic arguments to prove this result.

2 Existence of Non-unique Triangle Sequences

In this section we show, using arguments based on the geometry of the triangle iteration, that there exist infinite sets of points that all have the same infinite triangle sequence. We begin with some notation to describe the partition triangles.

Definition 1 *Define* $T^{-n}_{a_1 \dots a_n}(\alpha, \beta)$ *to be the unique point* $(x, y) \in \triangle_{a_1, \dots, a_n}$ *such that* $T^n(x, y) = (\alpha, \beta)$.

Proposition 1 *Given any finite triangle sequence* $(a_1, \dots, a_n)$ *there exists a line segment such that all points on that segment have the sequence* $(a_1, \dots, a_n)$.

Proof:

The set $T^{-n}_{a_1 \dots a_n}(I)$, where I is the interval $(0, 1)$, is precisely the set of all points with triangle sequence equal to $(a_1, \dots, a_n)$.

QED

Definition 2 *For any finite sequence of nonnegative integers,* $a_1 \dots a_n$, *let* $d(a_1, \dots a_n) = min\{length(T^{-n}_{a_1 \dots a_n}(I)),\ \ length(T^{-n}_{a_1 \dots a_{n-1}, a_n+1}(I))\}$, *where* I *is the interval* $(0, 1)$.

In the figure below, we have

$$\begin{array}{lcl lcl}
A & = & T^{-n}_{a_1 \dots a_n}(I) & a & = & T^{-n}_{a_1 \dots a_n+1}(1,0) \\
B & = & \underline{T^{-n}_{a_1 \dots a_n+1}(I) \qquad\qquad\qquad} & b & = & T^{-n}_{a_1 \dots a_n}(1,0) \\
C & = & \overline{T^{-n}_{a_1 \dots a_n+1}(1,0)\ T^{-n}_{a_1 \dots a_n}(1,0)} & c & = & T^{-n}_{a_1 \dots a_n}(0,0) \\
m & = & T^{-n}_{a_1 \dots a_n+1}(\frac{1}{l+1}, 0) & n & = & T^{-n}_{a_1 \dots a_n+1}(x, 0)
\end{array}$$

Note: $0 < \frac{1}{l+1} < x < 1$

Figure 1: Triangle created by the consecutive partition lines $T^{-n}_{a_1 \dots a_n}(I)$ and $T^{-n}_{a_1 \dots a_n+1}(I)$.

Lemma 1 *For any small $\epsilon > 0$ and any sequence $a_1 \dots a_n$ of nonnegative integers, there exists a nonnegative integer l such that $d(a_1 \dots a_n, l) \geq d(a_1 \dots a_n) - \epsilon$.*

Proof:

<u>Case I:</u> Consider the case where the angle at c is obtuse. Then all line segments from b to a point on B will be longer than A. Since the endpoints of consecutive partition lines are getting closer to c, we have:

$$\begin{aligned} d(a_1 \dots a_n, l) &= length(T^{-(n+1)}_{a_1 \dots a_n, l+1}(I)) \\ &\geq length(T^{-n}_{a_1 \dots a_n}(I)) \\ &\geq d(a_1 \dots a_n) - \epsilon \end{aligned}$$

Therefore the lemma holds when c is obtuse.

<u>Case II:</u> Consider the case where the angle at c is acute. Consider a line segment, N, in $\triangle_{a_1 \dots a_n}$ with the following properties:

(i) One endpoint is at $T^{-n}_{a_1 \dots a_n}(1, 0)$ (point b in Figure 1).
(ii) The other endpoint is at $T^{-n}_{a_1 \dots a_n+1}(x, 0)$ for some $x \in (1, 0)$ (on line B in Figure 1).
(iii) The length of the line segment is at least $length(T^{-n}_{a_1 \dots a_n}(I)) - \epsilon$.
(iv) Given any m on the segment $\overline{cn}$, $M > N$.

Properties (i) and (ii) will clearly be able to be satisfied. Since our line segment, N, gets longer as the endpoint n approaches c, we know that we can satisfy property (iii). By the above reasoning we also know that we will satisy property (iv) when m gets sufficiently close to c.

Choose $l \in \mathbb{Z}^+$ so that $\frac{1}{l+1} \leq x$. Then $T^{-n}_{a_1 \dots a_n+1}(\frac{1}{l+1}, 0)$, (point m in Figure 1) lies between c and n. So the line segment from b to m has length at least $length(T^{-n}_{a_1 \dots a_n}(I)) - \epsilon = A - \epsilon$. But this line segment is also $T^{-(n+1)}_{a_1 \dots a_n, l}(I)$.

Then, since $length(T^{-(n+1)}_{a_1 \dots a_n, l+1}(I)) > length(T^{-(n+1)}_{a_1 \dots a_n, l}(I))$, we have:

$$\begin{aligned} d(a_1 \dots a_n, l) &= length(T^{-(n+1)}_{a_1 \dots a_n, l}(I)) \\ &\geq length(T^{-n}_{a_1 \dots a_n}(I)) - \epsilon \\ &\geq d(a_1 \dots a_n) - \epsilon \end{aligned}$$

QED

Theorem 1 *There exist distinct points* (α_1, β_1) *and* (α_2, β_2) *in* $\triangle$ *with identical infinite triangle sequences.*

Proof:

Consider a triangle sequence constructed in the following manner:
Choose a_1 so that $d(a_1) \geq 1 - \frac{1}{4}$.
Choose a_2 so that $d(a_1, a_2) \geq d(a_1) - \frac{1}{8} \geq 1 - \frac{1}{4} - \frac{1}{8}$.
Choose a_n so that $d(a_1 \ldots a_n) \geq d(a_1 \ldots a_{n-1}) - \frac{1}{2^{n+1}}$ for $n \geq 2$.

Our lemma guarantees that a sequence may be constructed in this way.

Then for all n, $d(a_1 \ldots a_n) \geq 1 - \frac{1}{4} - \frac{1}{8} - \ldots - \frac{1}{2^{n+1}} > \frac{1}{2}$. Therefore there is some line segment of length $\frac{1}{2}$ which is inside any triangle $\triangle_{a_1 \ldots a_n}$. Thus every point on this line segment has the infinite triangle sequence $(a_1, a_2, \ldots)$.

QED

3 Properties of Cross Products of Approximation Vectors

3.1 Motivation

We know that the approximation vectors C_n approach the plane given by $x + \alpha y + \beta z = 0$. Thus, it is reasonable to suppose that the cross-products $C_n \times C_{n+1}$ would approach the vector $(1, \alpha, \beta)$, which is normal to the plane. Since the approximation vectors are completely determined by the sequence, it is easy to see that the relation

$$\lim_{n \to \infty} \frac{C_n \times C_{n+1}}{||C_n \times C_{n+1}||} = (1, \alpha, \beta),$$

implies that the pair (α, β) is completely determined by the sequence and thus that the sequence is unique. The above limit relation is, in fact, equivalent to the uniqueness of the sequence.

As was shown in section 2, since a triangle sequence does not necessarily uniquely determine the pair (α, β), it is not always true that the cross product approaches the normal vector $(1, \alpha, \beta)$. However, it is still useful to investigate the properties of these cross products as an approach to answering questions of uniqueness.

3.2 Notation and Results

Definition 3 *Let* $X_n = C_n \times C_{n+1}$. *Set* $X_n = (x_n, y_n, z_n)$.

Lemma 2 $X_{n+1} = X_{n-2} + a_{n+1}X_{n-1} + X_n$

Proof:

$$\begin{aligned}
X_{n+1} &= C_{n+1} \times C_{n+2} \\
&= C_{n+1} \times (C_{n-1} - C_n - a_{n+2}C_{n+1}) \\
&= C_{n+1} \times C_{n-1} + C_n \times C_{n+1} \\
&= (C_{n-2} - C_{n-1} - a_{n+1}C_n) \times C_{n-1} + C_n \times C_{n+1} \\
&= C_{n-2} \times C_{n-1} + a_{n+1}C_{n-1} \times C_n + C_n \times C_{n+1} \\
&= X_{n-2} + a_{n+1}X_{n-1} + X_n
\end{aligned}$$

QED

Lemma 3 $X_n \times X_{n+1} = C_{n+1}$

Proof:
We know that

$$\begin{aligned}
x_n &= q_n r_{n+1} - q_{n+1} r_n \\
y_n &= r_n p_{n+1} - r_{n+1} p_n \\
z_n &= p_n q_{n+1} - p_{n+1} q_n
\end{aligned}$$

Let us refer to the third component of $X_n \times X_{n+1}$ as w_{n+1}. Then we know that

$$\begin{aligned}
w_{n+1} &= x_n y_{n+1} - y_n x_{n+1} \\
&= (q_n r_{n+1} - q_{n+1} r_n)(r_{n+1} p_{n+2} - r_{n+2} p_{n+1}) - (r_n p_{n+1} - r_{n+1} p_n)(q_{n+1} r_{n+2} - q_{n+2} r_{n+1}) \\
&= r_{n+1}(p_{n+2} q_n r_{n+1} - q_n p_{n+1} r_{n+2} - r_n q_{n+1} p_{n+2}) + r_n r_{n+2} q_{n+1} p_{n+1} \\
&\quad - r_{n+1}(r_{n+1} p_n q_{n+2} - p_n q_{n+1} r_{n+2} - r_n q_{n+2} p_{n+1}) - r_n r_{n+2} q_{n+1} p_{n+1} \\
&= r_{n+1}(p_{n+2} q_n r_{n+1} + p_n q_{n+1} r_{n+2} + r_n q_{n+2} p_{n+1} - r_{n+1} p_n q_{n+2} - q_n p_{n+1} r_{n+2} - r_n q_{n+1} p_{n+2})
\end{aligned}$$

Therefore, since $\det(C_n, C_{n+1}, C_{n+2}) = 1$, $w_{n+1} = r_{n+1}$.

Now, since X_n and X_{n+1} are both perpendicular to C_{n+1}, we know that $X_n \times X_{n+1} = kC_{n+1}$ for some k. However, since $w_{n+1} = r_{n+1}$, we know that $k = 1$ and hence that $X_n \times X_{n+1} = C_{n+1}$.

QED

Corollary 1 $\det(X_{n-2}, X_{n-1}, X_n) = 1$

Proof:

$$\begin{aligned}\det(X_{n-2}, X_{n-1}, X_n) &= X_{n-2} \cdot (X_{n-1} \times X_n) \\ &= X_{n-2} \cdot C_n \\ &= (C_{n-2} \times C_{n-1}) \cdot C_n \\ &= \det(C_{n-2}, C_{n-1}, C_n) \\ &= 1\end{aligned}$$

QED

Corollary 2 $|\frac{y_{n-1}}{x_{n-1}} - \frac{y_n}{x_n}| = \frac{|r_n|}{x_n x_{n-1}}$

Proof:
From Lemma 3 we know that

$$|x_n y_{n-1} - y_n x_{n-1}| = |r_n|.$$

Dividing this expression by $x_n x_{n-1}$ yields

$$\left|\frac{y_{n-1}}{x_{n-1}} - \frac{y_n}{x_n}\right| = \frac{|r_n|}{x_n x_{n-1}}.$$

QED

Definition 4 *Let α_n be defined so that $d_{n-3} - d_{n-2} - \alpha_n d_{n-1} = 0$.*

Definition 5 *Let $\tilde{C_n} = C_{n-3} - C_{n-2} - \alpha_n C_{n-1}$.*

Definition 6 *Let $\epsilon_n = \alpha_n - a_n$.*

From these definitions it should be clear that

$$a_n \leq \alpha_n < a_n + 1$$

$$0 \leq \epsilon_n < 1$$

$$\tilde{C_n} \cdot (1, \alpha, \beta) = 0$$

Additionally, if we let,

$$\tilde{C_n} \times \tilde{C}_{n+1} \equiv \tilde{X_n} \equiv (\tilde{x}_n, \tilde{y}_n, \tilde{z}_n).$$

Then,

$$\alpha = \frac{\tilde{y}_n}{\tilde{x}_n}$$

and

$$\beta = \frac{\tilde{z}_n}{\tilde{x}_n}.$$

Lemma 4 $\tilde{X}_n = X_n + \alpha_n \epsilon_n X_{n-1} + \epsilon_n X_{n-2}$

Proof:

$$\begin{aligned}
\tilde{X}_n &= \tilde{C}_n \times \tilde{C}_{n+1} \\
&= (C_{n-3} - C_{n-2} - \alpha_n C_{n-1}) \times (C_{n-2} - C_{n-1} - \alpha_{n+1} C_n) \\
&= C_{n-3} \times C_{n-2} - C_{n-3} \times C_{n-1} - \alpha_{n+1} C_{n-3} \times C_n + C_{n-2} \times C_{n-1} \\
&\quad \alpha_{n+1} C_{n-2} \times C_n - \alpha_n C_{n-1} \times C_{n-2} + \alpha_n \alpha_{n+1} C_{n-1} \times C_n
\end{aligned}$$

We now compute the following:

$$\begin{aligned}
C_{n-3} \times C_{n-1} &= C_{n-3} \times (C_{n-4} - C_{n-3} - a_{n-1} C_{n-2}) \\
&= -X_{n-4} - a_{n-1} X_{n-3}
\end{aligned}$$

$$\begin{aligned}
C_{n-3} \times C_n &= C_{n-3} \times (C_{n-3} - C_{n-2} - a_n C_{n-1}) \\
&= -X_{n-3} - a_n (C_{n-3} \times C_{n-1}) \\
&= -X_{n-3} + a_n X_{n-4} + a_n a_{n-1} X_{n-3}
\end{aligned}$$

$$\begin{aligned}
C_{n-2} \times C_n &= C_{n-2} \times (C_{n-3} - C_{n-2} - a_n C_{n-1}) \\
&= -X_{n-3} - a_n X_{n-2}
\end{aligned}$$

Therefore we have that,

$$\begin{aligned}
\tilde{X}_n &= X_{n-3} + X_{n-4} + a_{n-1} X_{n-3} - \alpha_{n+1}(-X_{n-3} + a_n X_{n-4} + a_n a_{n-1} X_{n-3} \\
&\quad X_{n-2} + \alpha_{n+1}(-X_{n-3} - a_n X_{n-2}) + \alpha_n X_{n-2} + \alpha_n \alpha_{n+1} X_{n-1} \\
&= \alpha_n \alpha_{n+1} X_{n-1} + (\alpha_n + 1 - \alpha_{n+1} a_n) X_{n-2} \\
&\quad + (1 + a_{n-1} - \alpha_{n+1} a_n a_{n-1}) X_{n-3} + (1 - \alpha_{n+1} a_n) X_{n-4} \\
&= (\alpha_n \alpha_{n+1} X_{n-1} - (\alpha_{n+1} a_n)(X_{n-2} + a_{n-1} X_{n-3} + X_{n-4}) \\
&\quad + (X_{n-2} + a_{n-1} X_{n-3} + X_{n-4}) + a_n X_{n-2} + X_{n-3} + \epsilon_n X_{n-2} \\
&= \alpha_{n+1} \epsilon_n X_{n-1} + X_{n-1} + a_n X_{n-2} + X_{n-3} + \epsilon_n X_{n-2} \\
&= X_n + \alpha_{n+1} \epsilon_n X_{n-1} + \epsilon_n X_{n-2}
\end{aligned}$$

QED

Lemma 5 $|\frac{y_{n-1}}{x_{n-1}} - \alpha| \leq \frac{|r_n| + |r_{n-1}|}{x_n x_{n-1}}$

Proof:
We know that,

$$\alpha = \frac{\tilde{y}_n}{\tilde{x}_n} = \frac{y_n + \alpha_{n+1} \epsilon_n y_{n-1} + \epsilon_n y_{n-2}}{x_n + \alpha_{n+1} \epsilon_n x_{n-1} + \epsilon_n x_{n-2}}.$$

Hence,

$$\begin{aligned}\left|\frac{y_{n-1}}{x_{n-1}}-\alpha\right| &= \frac{y_{n-1}}{x_{n-1}}-\frac{y_n+\alpha_{n+1}\epsilon_n y_{n-1}+\epsilon_n y_{n-2}}{x_n+\alpha_{n+1}\epsilon_n x_{n-1}+\epsilon_n x_{n-2}} \\ &= \frac{(x_n y_{n-1}-y_n x_{n-1})+(y_{n-1}x_{n-2}-x_{n-1}y_{n-2})\epsilon_n}{x_{n-1}(x_n+\alpha_{n+1}\epsilon_n x_{n-1}+\epsilon_n x_{n-2})}\end{aligned}$$

Therefore, since $\alpha_n \geq 0$, $\epsilon_n \geq 0$ and $x_n \geq 0$, then

$$\begin{aligned}\left|\frac{y_{n-1}}{x_{n-1}}-\alpha\right| &\leq \frac{|r_{n-1}\epsilon_n - r_n|}{x_n x_{n-1}} \\ \left|\frac{y_{n-1}}{x_{n-1}}-\alpha\right| &\leq \frac{|r_n|+|r_{n-1}|}{x_n x_{n-1}}\end{aligned}$$

QED

Lemma 6 $|\frac{z_{n-1}}{x_{n-1}}-\beta| \leq \frac{|q_n|+|q_{n-1}|}{x_n x_{n-1}}$

Proof:
The proof of this lemma is similar to the proof of the previous lemma.

QED

Lemma 7 *If* $\lim_{n\to\infty}\frac{|r_n|}{x_n x_{n-1}} = 0$ *and* $\lim_{n\to\infty}\frac{|q_n|}{x_n x_{n-1}} = 0$ *then the triangle sequence is unique.*

Proof:
Since $\lim_{n\to\infty}\frac{|r_n|}{x_n x_{n-1}} = 0$, we know that

$$\begin{aligned}\lim_{n\to\infty}\frac{|r_{n-1}|}{x_{n-1}x_{n-2}} &= 0 \\ \Rightarrow \lim_{n\to\infty}\frac{|r_{n-1}|}{x_n x_{n-1}} &= 0 \\ \Rightarrow \lim_{n\to\infty}\frac{|r_n|+|r_{n-1}|}{x_n x_{n-1}} &= 0\end{aligned}$$

Similarily, since $\lim_{n\to\infty}\frac{|q_n|}{x_n x_{n-1}} = 0$, we know that

$$\begin{aligned}\lim_{n\to\infty}\frac{|q_{n-1}|}{x_{n-1}x_{n-2}} &= 0 \\ \Rightarrow \lim_{n\to\infty}\frac{|q_{n-1}|}{x_n x_{n-1}} &= 0 \\ \Rightarrow \lim_{n\to\infty}\frac{|q_n|+|q_{n-1}|}{x_n x_{n-1}} &= 0\end{aligned}$$

Therefore by the two previous lemmas,

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \alpha$$
$$\lim_{n\to\infty} \frac{z_n}{x_n} = \beta$$

Hence, the triangle sequence is unique.

QED

4 Uniqueness of Certain Bounded Triangle Sequences

4.1 Review of Key Ideas

In section 3, we demonstrated that the relations

$$\lim_{n\to\infty} \frac{|r_n|}{x_n x_{n-1}} = 0$$

$$\lim_{n\to\infty} \frac{|q_n|}{x_n x_{n-1}} = 0$$

are equivalent to uniqueness. Now we will use this formulation to prove the uniqueness of certain bounded infinte triangle sequences in the two main theorems of this section.

4.2 Proof of the Theorem

We prove the following theorems by finding lower bounds on the growth of $x_n x_{n-1}$ and upper bounds on the growth of r_n and q_n in order to show that the appropriate limit relations hold.

Theorem 2 *The infinite triangle sequence given by $a_i = A$ for all i is unique.*

Proof:

First we shall bound r_n and q_n from above. We know that,

$$r_{n+1} = r_{n-2} - r_{n-1} - a_{n+1} r_n$$

$$|r_{n+1}| \leq |r_{n-1}| + |r_{n-1}| + A|r_n|$$

Additionally,

$$q_{n+1} = q_{n-2} - q_{n-1} - a_{n+1} q_n$$

$$|q_{n+1}| \leq |q_{n-1}| + |q_{n-1}| + A|q_n|$$

Consider the case where $A \geq 1$. We will show that $|r_n| \leq (A+1)^n$.

$$\begin{aligned} |r_0| &= 1 && \leq (A+1)^0 \\ |r_1| &= A && \leq (A+1)^1 \\ |r_2| &= A^2 - 1 && \leq (A+1)^2 \end{aligned}$$

Suppose that $|r_k| \leq (A+1)^k$ for all $k \leq n$. Then,

$$\begin{aligned} |r_{n+1}| &\leq |r_{n-2}| + |r_{n-1}| + A|r_n| \\ &\leq (A+1)^{n-2} + (A+1)^{n-1} + A(A+1)^n \\ &= (A+1)^n((A+1)^{-2} + (A+1)^{-1} + A) \\ &\leq (A+1)^{n+1} \end{aligned}$$

Note that the last of the above steps is justified since $A \geq 1$ implies that $(A+1)^{-2} + (A+1)^{-1} \leq 1$. Therefore, by induction, $|r_n| \leq (A+1)^n$ for all n.

Similarily we will show that $|q_n| \leq (A+1)^n$.

$$\begin{aligned} |q_0| &= 0 && \leq (A+1)^0 \\ |q_1| &= 1 && \leq (A+1)^1 \\ |q_2| &= A + 1 && \leq (A+1)^2 \end{aligned}$$

Suppose that $|q_k| \leq (A+1)^k$ for all $k \leq n$. Then,

$$\begin{aligned} |q_{n+1}| &\leq |q_{n-2}| + |q_{n-1}| + A|q_n| \\ &\leq (A+1)^{n-2} + (A+1)^{n-1} + A(A+1)^n \\ &= (A+1)^n((A+1)^{-2} + (A+1)^{-1} + A) \\ &\leq (A+1)^{n+1} \end{aligned}$$

Therefore, by induction, $|q_n| \leq (A+1)^n$ for all n.

Consider the case where $A = 0$. We will show that $|r_n| \leq (A + \frac{3}{2})^n = (\frac{3}{2})^n$.

$$\begin{aligned} |r_0| &= 1 \leq \left(\frac{3}{2}\right)^0 \\ |r_1| &= 0 \leq \left(\frac{3}{2}\right)^1 \\ |r_2| &= 1 \leq \left(\frac{3}{2}\right)^2 \end{aligned}$$

Suppose that $|r_k| \leq (\frac{3}{2})^k$ for all $k \leq n$. Then,

$$\begin{aligned}
|r_{n+1}| &\leq |r_{n-2}| + |r_{n-1}| \\
&\leq \left(\frac{3}{2}\right)^{n-2} + \left(\frac{3}{2}\right)^{n-1} \\
&= \left(\frac{3}{2}\right)^{n} \left(\left(\frac{3}{2}\right)^{-2} + \left(\frac{3}{2}\right)^{-1}\right) \\
&\leq \left(\frac{3}{2}\right)^{n+1}
\end{aligned}$$

Therefore, by induction, $|r_n| \leq (\frac{3}{2})^n$ for all n.

Similarily, we will show that $|q_n| \leq (A + \frac{3}{2})^n = (\frac{3}{2})^n$.

$$\begin{aligned}
|q_0| &= 0 \leq \left(\frac{3}{2}\right)^{0} \\
|q_1| &= 1 \leq \left(\frac{3}{2}\right)^{1} \\
|q_2| &= 1 \leq \left(\frac{3}{2}\right)^{2}
\end{aligned}$$

Suppose that $|q_k| \leq (\frac{3}{2})^k$ for all $k \leq n$. Then,

$$\begin{aligned}
|q_{n+1}| &\leq |q_{n-2}| + |q_{n-1}| \\
&\leq \left(\frac{3}{2}\right)^{n-2} + \left(\frac{3}{2}\right)^{n-1} \\
&= \left(\frac{3}{2}\right)^{n} \left(\left(\frac{3}{2}\right)^{-2} + \left(\frac{3}{2}\right)^{-1}\right) \\
&\leq \left(\frac{3}{2}\right)^{n+1}
\end{aligned}$$

Therefore, by induction, $|q_n| \leq (\frac{3}{2})^n$ for all n.

This means that for all $A \geq 0$ we have that

$$\begin{aligned}
|r_{n+1}| &\leq \left(A + \frac{3}{2}\right)^{n+1} \\
|q_{n+1}| &\leq \left(A + \frac{3}{2}\right)^{n+1}
\end{aligned}$$

Next we will bound x_n from below. We will show that $x_n \geq C(A+2)^{n/2}$.

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= 1 + A \\
x_2 &= 2 + 2A \\
x_3 &= 3 + 3A + A^2
\end{aligned}$$

Therefore,

$$x_2 \geq (A+2)x_0$$
$$x_3 \geq (A+2)x_1$$

Suppose that

$$x_k \geq (A+2)x_{k-2}$$
$$x_{k+1} \geq (A+2)x_{k-1}$$

We will show that $x_{k+2} \geq (A+2)x_k$. Since the x_n are increasing, we know that,

$$\begin{aligned}
x_{k+2} &= x_{k+1} + Ax_k + x_{k-1} \\
&= (x_k + Ax_{k-1} + x_{k-2}) + Ax_k + x_{k-1} \\
&= (A+1)x_k + (x_{k-1} + Ax_{k-1} + x_{k-2}) \\
&\geq (A+1)x_k + (x_{k-1} + Ax_{k-2} + x_{k-3}) \\
&= (A+1)x_k + x_k \\
&= (A+2)x_k
\end{aligned}$$

Therefore, by induction we have that $x_{n+2} \geq (A+2)x_n$ for all n. This implies that $x_n \geq C(A+2)^{n/2}$ for all n.

Finally we will show that $\frac{|r_n|}{x_n x_{n-1}}$ and $\frac{|q_n|}{x_n x_{n-1}}$ approach 0 as n grows without bound.

Recall that,

$$|r_n| \leq \left(A + \frac{3}{2}\right)^n$$
$$|q_n| \leq \left(A + \frac{3}{2}\right)^n$$
$$x_n \geq C(A+2)^{n/2}$$
$$x_{n-1} \geq C(A+2)^{(n/2-1/2)} \geq D(A+2)^{n/2}$$

Therefore, we have that

$$\lim_{n\to\infty} \frac{|r_n|}{x_n x_{n-1}} \leq \lim_{n\to\infty} \frac{(A+\frac{3}{2})^n}{CD(A+2)^n} = 0$$
$$\lim_{n\to\infty} \frac{|q_n|}{x_n x_{n-1}} \leq \lim_{n\to\infty} \frac{(A+\frac{3}{2})^n}{CD(A+2)^n} = 0$$

Thus, by Lemma 7, the triangle sequence $a_i = A$ is unique for all $A \geq 0$.

QED

This establishes uniqueness for the very particular case where $a_i = A$ for all i. Although we have made only partial progress toward an extension of this method, we do believe that it can be extended to prove uniqueness of all bounded triangle sequences.

5 Relationships Between Cross Product Vectors and Vertices of Partition Triangles

5.1 Overview

In order to understand the uniqueness question through the geometry of the iteration, we need to investigate the partition triangles $\triangle_{a_1,\dots,a_n}$. In particular, we need to study the vertices of these triangles. In the following section, we begin by stating and proving a lemma that relates the vertices of $\triangle_{a_1,\dots,a_n}$ to those of $\triangle_{a_n,\dots,a_1}$. This result is interesting in its own right, but it is most useful to us as a tool in proving the main theorem of the section. This main theorem shows that there is a clean relation between the vertices of $\triangle_{a_1,\dots,a_n}$ and the coordinates of the vectors X_n associated with the sequence $(a_1,\dots,a_n)$. Finally, as a corollary to this theorem, we will construct a formula for the area of $\triangle_{a_1,\dots,a_n}$.

5.2 Proofs of the Relations

The main theorem of this paper concerns a direct relationship between the vertices of the partition triangles and the coordinates of the cross product vectors in three dimensions.

Lemma 8 *(Reversing)*

Let $\triangle_{a_1,a_2,\dots,a_n}$ be the set of points whose triangle sequences have first n terms equal to $(a_1,a_2,\dots,a_n)$. Suppose the vertices of $\triangle_{a_1,a_2,\dots,a_n}$ are written as

$$T^{-n}_{a_1,a_2,\dots,a_n}(0,0) = \left(\frac{y_{00}}{x_{00}}, \frac{z_{00}}{x_{00}}\right)$$

$$T^{-n}_{a_1,a_2,\dots,a_n}(1,0) = \left(\frac{y_{10}}{x_{10}}, \frac{z_{10}}{x_{10}}\right)$$

$$T^{-n}_{a_1,a_2,\dots,a_n}(1,1) = \left(\frac{y_{11}}{x_{11}}, \frac{z_{11}}{x_{11}}\right)$$

Then the vertices of $\triangle_{a_n,a_{n-1},\dots,a_1}$ are equal to

$$T^{-n}_{a_n,a_{n-1},\dots,a_1}(0,0) = \left(\frac{y_{00}}{y_{10}}, \frac{y_{11}-y_{10}}{y_{10}}\right)$$

$$T^{-n}_{a_n,a_{n-1},\dots,a_1}(1,0) = \left(\frac{x_{00}}{x_{10}}, \frac{x_{11}-x_{10}}{x_{10}}\right)$$

$$T^{-n}_{a_n,a_{n-1},\dots,a_1}(1,1) = \left(\frac{z_{00}+x_{00}}{z_{10}+x_{10}}, \frac{z_{11}-z_{10}+x_{11}-x_{10}}{z_{10}+x_{10}}\right)$$

Proof:

In order to prove this lemma, it is necessary to ensure that this notation for the vertices is well-defined. It is easy to see that the vertices of the partition triangles will always be of the form (u, v) where u and v are rational. Write (u, v) in the form $(\frac{y}{x}, \frac{z}{x})$ where x,y and z are integers and x is as small as possible. This will ensure that $(x, y, z) = 1$ (unless ,of course, either u or v is zero).

The proof of the reversing lemma will proceed by induction on n, the length of the sequence. We will begin by establishing the base cases where $n = 0$ and $n = 1$.

<u>Case $n = 0$</u>

It is natural to consider the case $n = 0$ to be the entire triangle. We will denote this triangle by $\triangle$. The vertices of $\triangle$ are

$$(0, 0)$$

$$(1, 0)$$

$$(1, 1)$$

It is easy to compute that

$$\left(\frac{0}{1}, \frac{1-1}{1}\right) = (0, 0)$$

$$\left(\frac{1}{1}, \frac{1-1}{1}\right) = (1, 0)$$

$$\left(\frac{0+1}{0+1}, \frac{1-0+1-1}{0+1}\right) = (1, 1)$$

This proves the lemma for the case $n = 0$, since $\triangle$ is the reverse of itself.

Case $n = 1$

The vertices of $\triangle_{a_1}$ are

$$(1, 0)$$

$$\left(\frac{1}{a_1 + 1}, \frac{1}{a_1 + 1}\right)$$

$$\left(\frac{1}{a_1 + 2}, \frac{1}{a_1 + 2}\right)$$

It is again easy to compute that

$$\left(\frac{1}{1}, \frac{1-1}{1}\right) = (1, 0)$$

$$\left(\frac{1}{a_1 + 1}, \frac{(a_1 + 2) - (a_1 + 1)}{a_1 + 1}\right) = \left(\frac{1}{a_1 + 1}, \frac{1}{a_1 + 1}\right)$$

$$\left(\frac{0 + 1}{1 + (a_1 + 1)}, \frac{1 - 1 + (a_1 + 2) - (a_1 + 1)}{1 + (a_1 + 1)}\right) = \left(\frac{1}{a_1 + 2}, \frac{1}{a_1 + 2}\right)$$

This proves the lemma for the case $n = 1$, since $\triangle_{a_1}$ is the reverse of itself.

Now we assume that the theorem holds true for all triangle sequences of length k with $0 \leq k \leq n-1$ and show that the theorem holds true for sequences of length n. The structure of the proof will be as follows. We will begin with the coordinates of the vertices of $\triangle_{a_2,\ldots,a_{n-1}}$ and construct the coordinates of the vertices of both $\triangle_{a_1,\ldots,a_n}$ and $\triangle_{a_n,\ldots,a_1}$. We will then demonstrate that the appropriate relation occurs between the coordinates of the vertices of the two triangles.

In order to construct the vertices, we need the following relation, which follows easily from the definition of the mapping T.

$$T_k^{-1}\left(\frac{y}{x}, \frac{z}{x}\right) = \left(\frac{x}{ky + z + x}, \frac{y}{ky + z + x}\right)$$

This notation refers to the preimage of the point $(\frac{y}{x}, \frac{z}{x})$ in triangle k.

We also need to check that the operations of reversing and applying T_k^{-1} preserve the reduced fractional form of the vertex coordinates. This means that we need to check that, under one of these operations, a pair $(\frac{y}{x}, \frac{z}{x})$ with $(x, y, z) = 1$ will produce another pair $(\frac{y'}{x'}, \frac{z'}{x'})$ with $(x', y', z') = 1$. We need a couple of propositions.

Proposition 2 *If $(x, y, z) = 1$, then $T_k^{-1}(\frac{y}{x}, \frac{z}{x}) = (\frac{x}{ky+x+z}, \frac{y}{ky+x+z})$ is in reduced fractional form.*

Proof:

Suppose $(\frac{x}{ky+x+z}, \frac{y}{ky+x+z})$ is not in reduced fractional form. Then there is an positive integer $m \neq 1$ such that $m|x$, $m|y$ and $m|(ky + x + z)$. This implies that $m|z$, which contradicts the assumption that $(x, y, z) = 1$. This proves the proposition.

QED

Proposition 3 *If the points*

$$\left(\frac{y_{00}}{x_{00}}, \frac{z_{00}}{x_{00}}\right)$$

$$\left(\frac{y_{10}}{x_{10}}, \frac{z_{10}}{x_{10}}\right)$$

$$\left(\frac{y_{11}}{x_{11}}, \frac{z_{11}}{x_{11}}\right)$$

are in reduced fractional form, then

$$\left(\frac{y_{00}}{y_{10}}, \frac{y_{11} - y_{10}}{y_{10}}\right)$$

$$\left(\frac{x_{00}}{x_{10}}, \frac{x_{11} - x_{10}}{x_{10}}\right)$$

$$\left(\frac{z_{00} + x_{00}}{z_{10} + x_{10}}, \frac{z_{11} - z_{10} + x_{11} - x_{10}}{z_{10} + x_{10}}\right)$$

are in reduced fractional form.

Proof:

It suffices to show that

$$(y_{00}, y_{10}, y_{11} - y_{10}) = (x_{00}, x_{10}, x_{11} - x_{10}) = (z_{00} + x_{00}, z_{10} + x_{10}, z_{11} - z_{10} + x_{11} - x_{10}) = 1$$

This is equivalent to showing that

$$(y_{00}, y_{10}, y_{11}) = (x_{00}, x_{10}, x_{11}) = (z_{00} + x_{00}, z_{10} + x_{10}, z_{11} + x_{11}) = 1$$

We claim that

$$\det \begin{pmatrix} x_{00} & x_{10} & x_{11} \\ y_{00} & y_{10} & y_{11} \\ z_{00} & z_{10} & z_{11} \end{pmatrix} = 1$$

This implies that

$$\det \begin{pmatrix} x_{00} & x_{10} & x_{11} \\ y_{00} & y_{10} & y_{11} \\ x_{00}+z_{10} & x_{01}+z_{01} & x_{11}+z_{11} \end{pmatrix} = 1$$

which proves the lemma.

To prove the claim, we use induction on n, the length of the sequence. We define the vertices of $\triangle_{a_1,\dots,a_k}$ in reduced fractional form as follows:

$$\left(\frac{y_{00}^{(k)}}{x_{00}^{(k)}}, \frac{z_{00}^{(k)}}{x_{00}^{(k)}}\right)$$

$$\left(\frac{y_{10}^{(k)}}{x_{10}^{(k)}}, \frac{z_{10}^{(k)}}{x_{10}^{(k)}}\right)$$

$$\left(\frac{y_{11}^{(k)}}{x_{11}^{(k)}}, \frac{z_{11}^{(k)}}{x_{11}^{(k)}}\right)$$

and let

$$M^{(k)} = \begin{pmatrix} x_{00}^{(k)} & x_{10}^{(k)} & x_{11}^{(k)} \\ y_{00}^{(k)} & y_{10}^{(k)} & y_{11}^{(k)} \\ z_{00}^{(k)} & z_{10}^{(k)} & z_{11}^{(k)} \end{pmatrix}$$

We start with the case k=0. The vertices of $\triangle$ are:

$$\left(\frac{0}{1}, \frac{0}{1}\right)$$

$$\left(\frac{1}{1}, \frac{0}{1}\right)$$

$$\left(\frac{1}{1}, \frac{1}{1}\right)$$

Then

$$\det M^{(0)} = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

Now, suppose that $\det M^{(n-1)} = 1$. The matrix for $T_{a_n^{-1}}$ is as follows:

$$[T_{a_n}^{-1}] = \begin{pmatrix} 1 & a_n & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus $\det[T_{a_n}^{-1}] = 1$ Since $M^{(n)} = [T_{a_n}^{-1}]M^{(n-1)}$ this implies that $\det M^{(n)} = 1$ This proves the claim, and thus completes the proof of the proposition.

QED

Now, we are ready to construct the proof of the reversing lemma. Let $\triangle_{a_2,\cdots,a_{n-1}}$ have vertices equal to

$$\left(\frac{y_{00}}{x_{00}}, \frac{z_{00}}{x_{00}}\right)$$

$$\left(\frac{y_{10}}{x_{10}}, \frac{z_{10}}{x_{10}}\right)$$

$$\left(\frac{y_{11}}{x_{11}}, \frac{z_{11}}{x_{11}}\right)$$

Now, if we apply the mapping $T_{a_1}^{-1}$ to each of these points, we recover the vertices of $\triangle_{a_1,\cdots,a_{n-1}}$, which are calculated to be:

$$\left(\frac{x_{00}}{a_1y_{00} + z_{00} + x_{00}}, \frac{y_{00}}{a_1y_{00} + z_{00} + x_{00}}\right)$$

$$\left(\frac{x_{10}}{a_1y_{10} + z_{10} + x_{10}}, \frac{y_{10}}{a_1y_{10} + z_{10} + x_{10}}\right)$$

$$\left(\frac{x_{11}}{a_1y_{11} + z_{11} + x_{11}}, \frac{y_{11}}{a_1y_{11} + z_{11} + x_{11}}\right)$$

These are the vertices of a triangle whose sequence has $n-1$ terms. We then apply the reversing lemma to the vertices to yield the vertices of $\triangle_{a_{n-1},\cdots,a_1}$, shown below.

$$\left(\frac{x_{00}}{x_{10}}, \frac{x_{11} - x_{10}}{x_{10}}\right)$$

$$\left(\frac{a_1y_{00} + z_{00} + x_{00}}{a_1y_{10} + z_{10} + x_{10}}, \frac{a_1y_{11} - a_1y_{10} + z_{11} - z_{10} + x_{11} - x_{10}}{a_1y_{10} + z_{10} + x_{10}}\right)$$

$$\left(\frac{y_{00} + a_1y_{00} + z_{00} + x_{00}}{y_{10} + a_1y_{10} + z_{10} + x_{10}}, \frac{y_{11} - y_{10} + a_1y_{11} - a_1y_{10} + z_{11} - z_{10} + x_{11} - x_{10}}{y_{10} + a_1y_{10} + z_{10} + x_{10}}\right)$$

Applying $T_{a_n}^{-1}$ to these points yields the vertices of $\triangle_{a_n,\dots,a_1}$:

$$\left(\frac{x_{10}}{a_n x_{00}+x_{11}}, \frac{x_{00}}{a_n x_{00}+x_{11}}\right)$$

$$\left(\frac{a_1 y_{10}+z_{10}+x_{10}}{a_n a_1 y_{00}+a_n z_{00}+a_n x_{00}+a_1 y_{11}+z_{11}+x_{11}}, \frac{a_1 y_{00}+z_{00}+x_{00}}{a_n a_1 y_{00}+a_n z_{00}+a_n x_{00}+a_1 y_{11}+z_{11}+x_{11}}\right)$$

$$\left(\frac{y_{10}+a_1 y_{10}+z_{10}+x_{10}}{a_n y_{00}+a_n a_1 y_{00}+a_n z_{00}+a_n x_{00}+y_{11}+a_1 y_{11}+z_{11}+x_{11}}, \frac{y_{00}+a_1 y_{00}+z_{00}+x_{00}}{a_n y_{00}+a_n a_1 y_{00}+a_n z_{00}+a_n x_{00}+y_{11}+a_1 y_{11}+z_{11}+x_{11}}\right)$$

Now we will similarly construct the vertices of $\triangle_{a_1,\dots,a_n}$.

First, we use the reversing lemma to calculate the vertices of $\triangle_{a_{n-1},\dots,a_2}$ to be:

$$\left(\frac{y_{00}}{y_{10}}, \frac{y_{11}-y_{10}}{y_{10}}\right)$$

$$\left(\frac{x_{00}}{x_{10}}, \frac{x_{11}-x_{10}}{x_{10}}\right)$$

$$\left(\frac{z_{00}+x_{00}}{z_{10}+x_{10}}, \frac{z_{11}-z_{10}+x_{11}+x_{10}}{z_{10}+x_{10}}\right)$$

Applying $T_{a_n}^{-1}$ yields the vertices of $\triangle_{a_n,\dots,a_2}$:

$$\left(\frac{y_{10}}{a_n y_{00}+y_{11}}, \frac{y_{00}}{a_n y_{00}+y_{11}}\right)$$

$$\left(\frac{x_{10}}{a_n x_{00}+x_{11}}, \frac{x_{00}}{a_n x_{00}+x_{11}}\right)$$

$$\left(\frac{z_{10}+x_{10}}{a_n z_{00}+a_n x_{00}+z_{11}+x_{11}}, \frac{z_{00}+x_{00}}{a_n z_{00}+a_n x_{00}+z_{11}+x_{11}}\right)$$

Now we apply the reversing lemma to calculate the vertices of $\triangle_{a_2,\dots,a_n}$:

$$\left(\frac{y_{10}}{x_{10}}, \frac{z_{10}}{x_{10}}\right)$$

$$\left(\frac{a_n y_{00}+y_{11}}{a_n x_{00}+x_{11}}, \frac{a_n z_{00}+z_{11}}{a_n x_{00}+x_{11}}\right)$$

$$\left(\frac{a_n y_{00}+y_{00}+y_{11}}{a_n x_{00}+x_{00}+x_{11}}, \frac{a_n z_{00}+z_{00}+z_{11}}{a_n x_{00}+x_{00}+x_{11}}\right)$$

Finally, we apply $T_{a_1}^{-1}$ to yield the vertices of $\triangle_{a_1,\dots,a_n}$:

$$\left(\frac{x_{10}}{a_1y_{10}+x_{10}+z_{10}}, \frac{y_{10}}{a_1y_{10}+x_{10}+z_{10}}\right)$$

$$\left(\frac{a_nx_{00}+x_{11}}{a_1a_ny_{00}+a_1y_{11}+a_nz_{00}+z_{11}+a_nx_{00}+x_{11}}, \frac{a_ny_{00}+y_{11}}{a_1a_ny_{00}+a_1y_{11}+a_nz_{00}+z_{11}+a_nx_{00}+x_{11}}\right)$$

$$\left(\frac{x_{00}+a_nx_{00}+x_{11}}{a_1y_{00}+a_1a_ny_{00}+a_1y_{11}+x_{00}+a_nx_{00}+x_{11}+z_{00}+a_nz_{00}+z_{11}}, \right.$$
$$\left. \frac{y_{00}+a_ny_{00}+y_{11}}{a_1y_{00}+a_1a_ny_{00}+a_1y_{11}+x_{00}+a_nx_{00}+x_{11}+z_{00}+a_nz_{00}+z_{11}}\right)$$

But, this is what you get if you apply the reversing lemma to $\triangle_{a_n,\dots,a_1}$. Thus the lemma is true for sequences of length n and is thus true by induction for all sequences.

QED

Now we are ready to prove the main theorem of this exposition:

Theorem 3 *Let $X_n = C_n \times C_{n+1} = (x_n, y_n, z_n)$. Then the vertices of $\triangle_{a_1,\dots,a_n}$ are given by:*

$$T_{a_1,\dots,a_n}^{-n}(0,0) = \left(\frac{y_{n-1}}{x_{n-1}}, \frac{z_{n-1}}{x_{n-1}}\right)$$

$$T_{a_1,\dots,a_n}^{-n}(0,1) = \left(\frac{y_n}{x_n}, \frac{z_n}{x_n}\right)$$

$$T_{a_1,\dots,a_n}^{-n}(1,1) = \left(\frac{y_{n-2}+y_n}{x_{n-2}+x_n}, \frac{z_{n-2}+z_n}{x_{n-2}+x_n}\right)$$

Proof:

This theorem is proven by induction on n, the number of terms in the sequence. We consider first the case $n = 1$.

The vertices of $\triangle_{a_1}$ are:

$$(1,0)$$
$$\left(\frac{1}{a_1+1}, \frac{1}{a_1+1}\right)$$
$$\left(\frac{1}{a_1+2}, \frac{1}{a_1+2}\right)$$

We also know that:

$$X_{-1} = (x_{-1}, y_{-1}, z_{-1}) = (1, 0, 0)$$

$$X_0 = (x_0, y_0, z_0) = (1, 1, 0)$$

$$X_1 = (x_1, y_1, z_1) = (a_1 + 1, 1, 1)$$

We check that:

$$\left(\frac{y_0}{x_0}, \frac{z_0}{x_0}\right) = \left(\frac{1}{1}, \frac{0}{1}\right) = (1, 0)$$

$$\left(\frac{y_1}{x_1}, \frac{z_1}{x_1}\right) = \left(\frac{1}{a_1 + 1}, \frac{1}{a_1 + 1}\right)$$

$$\left(\frac{y_{-1} + y_1}{x_{-1} + x_1}, \frac{z_{-1} + z_1}{x_{-1} + x_1}\right) = \left(\frac{0 + 1}{(a_1 + 1) + 1}, \frac{0 + 1}{(a_1 + 1) + 1}\right) = \left(\frac{1}{a_1 + 2}, \frac{1}{a_1 + 2}\right)$$

Now we assume that the theorem holds for the vertices of all triangles with sequences of length k with $1 \leq k \leq n - 1$. Suppose the vertices of $\triangle_{a_1,\dots,a_{n-1}}$ are

$$\left(\frac{y_{n-2}}{x_{n-2}}, \frac{z_{n-2}}{x_{n-2}}\right)$$

$$\left(\frac{y_{n-1}}{x_{n-1}}, \frac{z_{n-1}}{x_{n-1}}\right)$$

$$\left(\frac{y_{n-3} + y_{n-1}}{x_{n-3} + x_{n-1}}, \frac{z_{n-3} + z_{n-1}}{x_{n-3} + x_{n-1}}\right)$$

We will then construct the vertices of $\triangle_{a_1,\dots,a_n}$ by applying the reversing lemma and the map $T_{a_n}^{-1}$.

First, the reversing lemma gives us the vertices of $\triangle_{a_{n-1},\dots,a_1}$

$$\left(\frac{y_{n-2}}{y_{n-1}}, \frac{y_{n-3}}{y_{n-1}}\right)$$

$$\left(\frac{x_{n-2}}{x_{n-1}}, \frac{x_{n-3}}{x_{n-1}}\right)$$

$$\left(\frac{z_{n-2} + x_{n-2}}{z_{n-1} + x_{n-1}}, \frac{z_{n-3} + x_{n-3}}{z_{n-1} + x_{n-1}}\right)$$

Next, we apply the map $T_{a_n}^{-1}$ to give the vertices of $\triangle_{a_n,\dots,a_1}$

$$\left(\frac{y_{n-1}}{y_{n-1}+a_n y_{n-2}+y_{n-3}}, \frac{y_{n-2}}{y_{n-1}+a_n y_{n-2}+y_{n-3}}\right)$$

$$\left(\frac{x_{n-1}}{x_{n-1}+a_n x_{n-2}+x_{n-3}}, \frac{x_{n-2}}{x_{n-1}+a_n x_{n-2}+x_{n-3}}\right)$$

$$\left(\frac{x_{n-1}+z_{n-1}}{x_{n-1}+a_n x_{n-2}+x_{n-3}+z_{n-1}+a_n z_{n-2}+z_{n-3}}, \frac{x_{n-2}+z_{n-2}}{x_{n-1}+a_n x_{n-2}+x_{n-3}+z_{n-1}+a_n z_{n-2}+z_{n-3}}\right)$$

By using the recursion relation $X_n = X_{n-1}+a_n X_{n-2}+X_{n-3}$ we can simplify this

$$\left(\frac{y_{n-1}}{y_n}, \frac{y_{n-2}}{y_n}\right)$$

$$\left(\frac{x_{n-1}}{x_n}, \frac{x_{n-2}}{x_n}\right)$$

$$\left(\frac{x_{n-1}+z_{n-1}}{x_n+z_n}, \frac{x_{n-2}+z_{n-2}}{x_n+z_n}\right)$$

Finally, we apply the reversing lemma to these points to yield the vertices of $\triangle_{a_1,\dots,a_n}$. They are

$$\left(\frac{y_{n-1}}{x_{n-1}}, \frac{z_{n-1}}{x_{n-1}}\right)$$

$$\left(\frac{y_n}{x_n}, \frac{z_n}{x_n}\right)$$

$$\left(\frac{y_{n-2}+y_n}{x_{n-2}+x_n}, \frac{z_{n-2}+z_n}{x_{n-2}+x_n}\right)$$

This shows that the theorem is true for sequences of length n. This proves the theorem for all n.

QED

Corollary 3 *The area of* $\triangle_{a_1,\dots,a_n}$ *is equal to* $\frac{1}{2x_n x_{n-1}(x_n+x_{n-2})}$.

Proof:

We know from the previous theorem that the vertices of $\triangle_{a_1,\dots,a_n}$ are:

$$\left(\frac{y_{n-1}}{x_{n-1}}, \frac{z_{n-1}}{x_{n-1}}\right)$$

$$\left(\frac{y_n}{x_n}, \frac{z_n}{x_n}\right)$$

$$\left(\frac{y_{n-2}+y_n}{x_{n-2}+x_n}, \frac{z_{n-2}+z_n}{x_{n-2}+x_n}\right)$$

Therefore, we can calculate the area as one-half of the length of the cross-product of two edge vectors. The edge vectors are calculated to be:

$$E_1 = \left(\frac{y_{n-2}+y_n}{x_{n-2}+x_n} - \frac{y_n}{x_n}, \frac{z_{n-2}+z_n}{x_{n-2}+x_n} - \frac{z_n}{x_n}, 0\right)$$

$$E_2 = \left(\frac{y_{n-1}}{x_{n-1}} - \frac{y_n}{x_n}, \frac{z_{n-1}}{x_{n-1}} - \frac{z_n}{x_n}, 0\right)$$

Combining the fractions yields:

$$E_1 = \left(\frac{y_{n-2}x_n - x_{n-2}y_n}{x_n(x_{n-2}+x_n)}, \frac{z_{n-2}x_n - x_{n-2}z_n}{x_n(x_{n-2}+x_n)}, 0\right)$$

$$E_1 = \left(\frac{y_{n-1}x_n - x_{n-1}y_n}{x_n x_{n-1}}, \frac{z_{n-1}x_n - x_{n-1}z_n}{x_n x_{n-1}}, 0\right)$$

Now, we know that $y_{n-1}x_n - x_{n-1}y_n = -r_n$ and that $z_{n-1}x_n - x_{n-1}z_n = q_n$. We can also calculate the other two numerators by using the recursion relation

$$X_n = X_{n-1} + a_n X_{n-2} + X_{n-3}$$

$$\begin{aligned}
y_{n-2}x_n - x_{n-2}y_n &= y_{n-2}(x_{n-1} + a_n x_{n-2} + x_{n-3}) - x_{n-2}(y_{n-1} + a_n y_{n-2} + y_{n-3}) \\
&= (y_{n-2}x_{n-1} - x_{n-2}y_{n-1}) + (y_{n-2}x_{n-3} - x_{n-2}y_{n-3}) \\
&= -r_{n-1} + r_{n-2}
\end{aligned}$$

Similarly, $z_{n-2}x_n - x_{n-2}z_n = q_{n-1} - q_{n-2}$.

Thus we can rewrite E_1 and E_2 in the following way:

$$E_1 = \left(\frac{-r_{n-1} + r_{n-2}}{x_n(x_{n-2}+x_n)}, \frac{q_{n-1} - q_{n-2}}{x_n(x_{n-2}+x_n)}, 0\right)$$

$$E_2 = \left(\frac{-r_n}{x_n x_{n-1}}, \frac{q_n}{x_n x_{n-1}}, 0\right)$$

Therefore,

$$\begin{aligned}
E_1 \times E_2 &= \left(0, 0, \frac{(q_n)(-r_{n-1}+r_{n-2}) - (-r_n)(q_n - q_{n-2})}{x_n^2 x_{n-1}(x_n + x_{n-2})}\right) \\
&= \left(0, 0, \frac{(r_n q_{n-1} - q_n r_{n-1}) + (q_n r_{n-2} - r_n q_{n-2})}{x_n^2 x_{n-1}(x_n + x_{n-2})}\right) \\
&= \left(0, 0, \frac{(x_{n-1}) + ((q_{n-3} - q_{n-2} - a_n q_{n-1}) r_{n-2} - (r_{n-3} - r_{n-2} - a_n r_{n-1}) q_{n-2})}{x_n^2 x_{n-1}(x_n + x_{n-2})}\right) \\
&= \left(0, 0, \frac{x_{n-1} + (r_{n-2} q_{n-3} - r_{n-3} q_{n-2}) + a_n (r_{n-1} q_{n-2} - r_{n-2} q_{n-1})}{x_n^2 x_{n-1}(x_n + x_{n-2})}\right) \\
&= \left(0, 0, \frac{x_{n-1} + a_n x_{n-2} + x_{n-3}}{x_n^2 x_{n-1}(x_n + x_{n-2})}\right) \\
&= \left(0, 0, \frac{x_n}{x_n^2 x_{n-1}(x_n + x_{n-2})}\right) \\
&= \left(0, 0, \frac{1}{x_n x_{n-1}(x_n + x_{n-2})}\right)
\end{aligned}$$

Thus, $Area\ of \triangle_{a_1,\dots,a_n} = \frac{1}{2}||E_1 \times E_2|| = \frac{1}{2x_n x_{n-1}(x_n + x_{n-2})}$

QED

Corollary 4 *The set of all points (α, β) with triangle sequence equal to $(a_1, a_2, \dots)$ consists of at most a line segment.*

Proof:

It is easy to show that if you have any two points (α_1, β_1) and (α_2, β_2) that have the same triangle sequence, then every point on the line segment connecting them will also have that same triangle sequence. Now, suppose there are three non-collinear points that all have the same triangle sequence. Clearly all points on the triangular perimeter defined by these three points will have the same triangle sequence. Then, given any point in the interior of this region we can construct a line segment containing this point with endpoints on the perimeter. This implies that there exists a region with nonzero area in which every point has the same triangle sequence. However this contradicts corollary 3.

QED

6 Uniqueness of Triangle Sequences in Which the Integer C Appears Infinitely Often

This section uses a geometric argument to show that, in the case that any integer C appears infintely often in a triangle sequence $(a_1, a_2, \dots)$, the partition

Figure 2: Triangle corresponding to Lemma 9

triangles $\triangle_{a_1,\dots,a_n}$ converge to the unique point with that triangle sequence. We begin with a couple of definitions.

Definition 7 *For any vector,* $V = (v_1, v_2)$*, let* $\| V \| = \max(|v_1|, |v_2|)$*.*

Definition 8 *For any two points* A *and* B*, let* $\overrightarrow{AB}$ *be the vector from* A *to* B*.*

Definition 9 *Let the distance between two points A and B be defined as* $\| \overrightarrow{AB} \|$*.*

Lemma 9 *Let* $\triangle ABC$ *be a triangle and let* M *and* N *be two points such that* M *lies on the line segment between* B *and* N*. Then* $\| \overrightarrow{AM} \| \geq \| \overrightarrow{AN} \| \Rightarrow \| \overrightarrow{AM} \| \leq \| \overrightarrow{AB} \|$*.*

Proof:

Since B, M and N are colinear and since M lies between N and B, then $\overrightarrow{BM} = a\overrightarrow{MN}$ for some $a \geq 0$.

Let m_1 be the largest component of the vector $\overrightarrow{AM}$. Without loss of generality, assume let m_1 be non-negative. Then $\| \overrightarrow{AM} \| = m_1$. Now let x_1, n_1 and b_1 be the corresponding components of $\overrightarrow{MN}$, $\overrightarrow{AN}$ and $\overrightarrow{AB}$ respectively. Then,

$$\| \overrightarrow{AM} \| \geq \| \overrightarrow{AN} \| \Rightarrow m_1 \geq n_1$$

Therefore, since $\overrightarrow{AN} = \overrightarrow{AM} + \overrightarrow{MN}$, we know that $x_1 \leq 0$. Additionally, since $\overrightarrow{AB} = \overrightarrow{AM} - a\overrightarrow{MN}$, we know that $b_1 \geq m_1$. This implies that $\| \overrightarrow{AB} \| \geq \| \overrightarrow{AM} \|$.

QED

Lemma 10 *Let $\triangle ABC$ be a triangle with $\| \overrightarrow{AC} \| \leq \| \overrightarrow{AB} \|$, then for any point M on the line segment between B and C, $\| \overrightarrow{AM} \| \leq \| \overrightarrow{AB} \|$.*

Proof:

Case1: $\| \overrightarrow{AM} \| \geq \| \overrightarrow{AC} \|$

This case follows immediately by letting $N = C$ in the previous lemma.

Case2: $\| \overrightarrow{AM} \| < \| \overrightarrow{AC} \|$

Since $\| \overrightarrow{AC} \| \leq \| \overrightarrow{AB} \|$ and $\| \overrightarrow{AM} \| < \| \overrightarrow{AC} \|$, then $\| \overrightarrow{AM} \| \leq \| \overrightarrow{AB} \|$ as desired.

QED

Definition 10 *Let $s_0(n)$ be the length of the side connecting $T^{-n}_{a_1 \ldots a_n}(0,0)$ and $T^{-n}_{a_1 \ldots a_n}(1,0)$.*

Lemma 11 *For any $n \geq 3$ one of the following is true:*

$$
\begin{aligned}
s_0(n) &\leq s_0(n-1) \\
s_0(n) &\leq s_0(n-2) \\
s_0(n) &\leq s_0(n-3)
\end{aligned}
$$

Proof:

Let

$$
\begin{aligned}
A &= T^{-n}_{a_1 \ldots a_{n-1}}(1,0) \\
B &= T^{-n}_{a_1 \ldots a_{n-1}}(0,0) \\
C &= T^{-n}_{a_1 \ldots a_{n-1}}(1,1)
\end{aligned}
$$

Analogously let,

$$
\begin{aligned}
A' &= T^{-n}_{a_1 \ldots a_{n-2}}(1,0) \\
B' &= T^{-n}_{a_1 \ldots a_{n-2}}(0,0) \\
C' &= T^{-n}_{a_1 \ldots a_{n-2}}(1,1) \\
A'' &= T^{-n}_{a_1 \ldots a_{n-3}}(1,0) \\
B'' &= T^{-n}_{a_1 \ldots a_{n-3}}(0,0) \\
C'' &= T^{-n}_{a_1 \ldots a_{n-3}}(1,1)
\end{aligned}
$$

Where $T^{-n}_{a_1 \ldots a_0}$ is the identity transformation. Note that $B = A'$ and $B' = A''$.

Case 1: $\| \overrightarrow{AB} \| \geq \| \overrightarrow{AC} \|$

By Lemma 10,

$$s_0(n) \leq \| \overrightarrow{AB} \| = s_0(n-1)$$

Note that $s_0(n)$ is the length of a line segment connents to A to a point on the line segment between B and C.

Case 2: $\| \overrightarrow{AC} \| \geq \| \overrightarrow{AB} \|$ and $\| \overrightarrow{BC} \| \geq \| \overrightarrow{AC} \|$.

Since $\overrightarrow{BC}$ is the longest side of $\triangle ABC$ and $s_0(n)$ lies within $\triangle ABC$, then $s_0(n) \leq \| \overrightarrow{BC} \|$. Note that A and C are both points on the line segment connenting B' and C' and that C lies between B' and A. Therefore, by Lemma 9, since $\| BC \| \geq \| AC \|$,

$$s_0(n-2) = \| \overrightarrow{A'B'} \| \geq \| \overrightarrow{A'C} \| = \| \overrightarrow{BC} \| \geq s_0(n)$$

Case 3: $\| \overrightarrow{BC} \| \leq \| \overrightarrow{AC} \|$, $\| \overrightarrow{AB} \| \leq \| \overrightarrow{AC} \|$ and $\| \overrightarrow{B'A'} \| \geq \| \overrightarrow{B'C'} \|$.

Recall that the side between B' and C' contains the points A and C. Therefore,

$$s_0(n-2) = \| \overrightarrow{B'A'} \| \geq \| \overrightarrow{B'C'} \| \geq \| \overrightarrow{AC} \|$$

We know that $\overrightarrow{AC}$ is the longest side of $\triangle ABC$, in which $s_0(n)$ lies. Therefore,

$$s_0(n-2) \geq \| \overrightarrow{AC} \| \geq s_0(n)$$

Case 4: $\| \overrightarrow{BC} \| \leq \| \overrightarrow{AC} \|$, $\| \overrightarrow{AB} \| \leq \| \overrightarrow{AC} \|$ and $\| \overrightarrow{B'A'} \| \leq \| \overrightarrow{B'C'} \|$.

Note that A' and C' are points on the line segment connecting B'' and C'' with C' between B'' and A'. Therefore by Lemma 9, since $\| \overrightarrow{B'C'} \| \geq \| \overrightarrow{B'A'} \|$,

$$s_0(n-3) = \| \overrightarrow{A''B''} \| \geq \| \overrightarrow{A''C'} \| = \| \overrightarrow{B'C'} \|$$

Since $\overrightarrow{AC}$ is the longest side of $\triangle ABC$ and since the side between B' and C' contains the points A and C, then

$$s_0(n-3) \geq \| \overrightarrow{B'C'} \| \geq \| \overrightarrow{AC} \| \geq s_0(n)$$

QED

Definition 11 *Let* $F_n = \max(\frac{|r_{n-4}|}{x_{n-4}x_{n-5}}, \frac{|r_{n-3}|}{x_{n-3}x_{n-4}}, \frac{|r_{n-2}|}{x_{n-2}x_{n-3}})$.

Definition 12 *Let* $G_n = \max(\frac{|q_{n-4}|}{x_{n-4}x_{n-5}}, \frac{|q_{n-3}|}{x_{n-3}x_{n-4}}, \frac{|q_{n-2}|}{x_{n-2}x_{n-3}})$.

Lemma 12 *If* $a_n = C$ *and* $a_{n-1} > C$ *then* $\frac{|r_n|}{x_n x_{n-1}} \leq \frac{C^2+3C+2}{C^2+4C+4} F_n$.

Proof:

To prove this lemma, we will bound $|r_n|$ from above in terms of $\max(|r_{n-2}|, |r_{n-3}|, |r_{n-4}|)$ and bound $x_n x_{n-1}$ from below in terms of $x_{n-2}x_{n-3}$. We will then use these two bounds fact that $F_n \leq \frac{\max(|r_{n-2}|,|r_{n-3}|,|r_{n-4}|)}{x_{n-2}x_{n-3}}$ to bound $\frac{|r_n|}{x_n x_{n-1}}$ from above in terms of F_n.

Recall that

$$|r_n| = |r_{n-3} - r_{n-2} - a_n r_{n-1}|$$

By making the substitution, $r_{n-1} = r_{n-4} - r_{n-3} - a_{n-1}r_{n-2}$, we get,

$$|r_n| = |r_{n-3} - r_{n-2} - a_n r_{n-4} + a_n r_{n-3} + a_n a_{n-1} r_{n-2}| \tag{1}$$

By using the triangle inequality and that $a_n = C$, we get,

$$|r_n| \leq (Ca_{n-1} + 2C + 2)\max(|r_{n-2}|, |r_{n-3}|, |r_{n-4}|)$$

Additionally,

$$x_n x_{n-1} = (x_{n-3} + a_n x_{n-2} + x_{n-1})x_{n-1}$$

By making the substitution, $x_{n-1} = x_{n-4} + a_{n-1}x_{n-3} + x_{n-2}$, we get

$$\begin{aligned} x_n x_{n-1} = ((x_{n-4} + a_{n-1} + 1)x_{n-3} + (a_n + 1)x_{n-2}) \\ (x_{n-4} + a_{n-1}x_{n-3} + x_{n-2}) \end{aligned} \tag{2}$$

By dropping the x_{n-4} terms and using that $a_n = C$, we get

$$x_n x_{n-1} \geq ((a_{n-1} + 1)x_{n-3} + (C + 1)x_{n-2})(a_{n-1}x_{n-3} + x_{n-2})$$

By dropping the x_{n-3}^2 terms we get

$$x_n x_{n-1} \geq (Ca_{n-1} + 2a_{n-1} + C + 2)x_{n-2}x_{n-3}$$

Therefore,

$$\begin{aligned} \frac{|r_n|}{x_n x_{n-1}} &= \frac{(Ca_{n-1} + 2C + 2)\max(|r_{n-2}|, |r_{n-3}|, |r_{n-4}|)}{(Ca_{n-1} + 2a_{n-1} + C + 2)x_{n-2}x_{n-3}} \\ &\leq \frac{Ca_{n-1} + 2C + 2}{Ca_{n-1} + 2a_{n-1} + C + 2}F_n \end{aligned}$$

Since the derivative of the above expression with respect to a_{n-1} is always negative in the region $a_{n-1} \geq C + 1$, then the above expression achieves its maximum in the region $a_{n-1} \geq C + 1$ when $a_{n-1} = C + 1$. Therefore,

$$\frac{|r_n|}{x_n x_{n-1}} \leq \frac{C^2 + 3C + 2}{C^2 + 4C + 4}F_n$$

QED

Lemma 13 *If $a_n = C$ and $a_{n-1} > C$ then $\frac{|q_n|}{x_n x_{n-1}} \leq \frac{C^2+3C+2}{C^2+4C+4} G_n$*

Proof:

The proof holds for q_n because in the proof of the previous lemma, nothing is used about r_n except that it satisfies the recurrence, $r_n = r_{n-3} - r_{n-2} - a_n r_{n-1}$ which q_n satisfies as well.

QED

Lemma 14 *If $a_n = C$, $a_{n-1} > C$ and $a_{n+1} \geq C$, then $\frac{|r_{n+1}|}{x_{n+1} x_n} \leq \frac{C^2+3C+2}{C^2+4C+4} F_n$*

Proof:

By replacing n with $n+1$ in Equation 1, we get

$$|r_{n+1}| = |-a_{n+1} r_{n-3} + (a_{n+1}+1) r_{n-2} + (a_n a_{n+1} - 1) r_{n-1}|$$

By making the substitution, $r_{n-1} = r_{n-4} - r_{n-3} - a_{n-1} r_{n-2}$, we get,

$$\begin{aligned} |r_{n+1}| = |&(a_n a_{n+1} - 1) r_{n-4} + (1 - a_{n+1} - C a_{n+1}) r_{n-3} \\ &+ (a_{n+1} + 1 + a_{n-1} - C a_{n+1} a_{n-1}) r_{n-2}| \end{aligned} \quad (3)$$

By using the triangle inequality and that $a_n = C$, we get

$$|r_{n+1}| = (2Ca_{n+1} + a_{n-1} C a_{n+1} + 2a_{n+1} + a_{n-1} + 3) \max(|r_{n-2}|, |r_{n-3}|, |r_{n-4}|)$$

Additionally, by replacing n with $n+1$ in Equation 2, we get

$$x_{n+1} x_n = (x_{n-3} + (a_n+1) x_{n-2} + (a_{n+1}+1) x_{n-1})(x_{n-3} + a_n x_{n-2} + x_{n-1})$$

By making the substitution, $x_{n-1} = x_{n-4} + a_{n-1} x_{n-3} + x_{n-2}$ we get

$$\begin{aligned} x_{n+1} x_n = ((a_{n+1}+1) x_4 + (a_{n-1} + a_{n-1} a_{n+1} + 1) x_{n-3} + (a_{n+1} + a_n + 2) x_{n-2}) \\ (x_{n-4} + (a_{n-1}+1) x_{n-3} + (a_n+1) x_{n-2}) \end{aligned} \quad (4)$$

By dropping the x_{n-4} terms and using that $a_n = C$, we get

$$x_{n+1} x_n \geq ((a_{n-1} + a_{n-1} a_{n+1} + 1) x_{n-3} + (a_{n+1} + C + 2) x_{n-2})((a_{n-1}+1) x_{n-3} + (C+1) x_{n-2})$$

By dropping the x_{n-3}^2 terms we get

$$x_{n+1} x_n \geq (C a_{n-1} a_{n+1} + 2C a_{n-1} + C a_{n+1} + C^2 + 5C + 2a_{n-1} a_{n+1} + 3a_{n-1} + 2a_{n+1} + 5) x_{n-2} x_{n-3}$$

Therefore,

$$\frac{|r_n|}{x_{n+1} x_n} \leq \frac{2Ca_{n+1} + a_{n-1} C a_{n+1} + 2a_{n+1} + a_{n-1} + 3}{C a_{n-1} a_{n+1} + 2C a_{n-1} + C a_{n+1} + C^2 + 5C + 2a_{n-1} a_{n+1} + 3a_{n-1} + 2a_{n+1} + 5} F_n$$

Since the derivative of the above expression with respect to a_{n-1} is always negative when $a_{n-1} \geq C+1$ and $a_{n+1} \geq C$, then the above expression achieves its maximum in the region bounded by $a_{n-1} \geq C+1$ and $a_{n+1} \geq C$ when $a_{n-1} = C+1$. Therefore,

$$\frac{|r_n|}{x_{n+1}x_n} \leq \frac{3Ca_{n+1} + C^2a_{n+1} + C + 2a_{n+1} + 4}{C^2a_{n+1} + 3C^2 + 4Ca_{n+1} + 10C + 4a_{n+1} + 8}F_n$$

Since the derivative of the above expression with respect to a_{n+1} is always positive when $a_{n+1} \geq C$, we can replace the right hand side of the above inequality with its limit as $a_{n+1} \to \infty$. Therefore,

$$\frac{|r_n|}{x_{n+1}x_n} \leq \frac{C^2 + 3C + 2}{C^2 + 4C + 4}F_n$$

QED

Lemma 15 *If $a_n = C$, $a_{n-1} > C$ and $a_{n+1} \geq C$ then $\frac{|q_{n+1}|}{x_{n+1}x_n} \leq \frac{C^2+3C+2}{C^2+4C+4}G_n$*

Proof:

The proof of this lemma is identical to the proof of the previous lemma with all references to r_i replaced by references to q_i.

QED

Lemma 16 *If $a_n = C$, $a_{n-1} > C$, $a_{n+1} \geq C$ and $a_{n+2} \geq C$ then $\frac{|r_{n+2}|}{x_{n+2}x_{n+1}} \leq \frac{C^2+3C+2}{C^2+4C+4}F_n$*

Proof:

By replacing n with $n+1$ in Equation 3, we get

$$|r_{n+2}| = |(a_{n+2}a_{n+1} - 1)r_{n+3} - (a_{n+2} + a_{n+2}a_{n+1} - 1)r_{n-2} + (a_{n+2} + 1 - a_{n+2}a_{n+1}a_n + a_n)r_{n-1}|$$

By making the substitution, $r_{n-1} = r_{n-4} - r_{n-3} - a_{n-1}r_{n-2}$, we get,

$$|r_{n+2}| = |(a_{n+2} + 1 - a_{n+2}a_{n+1}C + C)r_{n-4} + (a_{n+2}a_{n+1} - a_{n+2} - 2 + a_{n+2}a_{n+1}C - C)r_{n-3} - (a_{n+2} + a_{n+2}a_{n+1} - 1 + a_{n+2}a_{n-1} + a_{n-1} - a_{n+2}a_{n+1}a_{n-1}C + Ca_{n-1})r_{n-2}|$$

By using the triangle inequality and that $a_n = C$, we get

$$|r_{n+2} \leq (a_{n+2}a_{n+1}a_{n-1}C + 2a_{n+2}a_{n+1}C + Ca_{n-1} + 2C + 2a_{n+2}a_{n+1} + a_{n+2}a_{n-1} + a_{n-1} + 3a_{n+2} + 4)\max(|r_{n-2}|, |r_{n-3}|, |r_{n-4}|) \quad (5)$$

Additionally, by replacing n with $n+1$ in Equation 4, we get

$$x_{n+2}x_{n+1} = ((a_{n+2}+1)x_{n-3}+(a_n a_{n+2}+a_n+1)x_{n-2}+(a_{n+1}+a_{n+2}+2)x_{n-1}) \\ (x_{n-3}+(a_n+1)x_{n-2}+(a_{n+1}+1)x_{n-1})$$

By making the substitution, $x_{n-1} = x_{n-4} + a_{n-1}x_{n-3} + x_{n-2}$, we get

$$x_{n+2}x_{n+1} = ((a_{n+1}+a_{n+2}+2)x_{n-4}+((a_{n+2}+1+a_{n+1}a_{n-1}+a_{n+2}a_{n-1}+2a_{n-1})x_{n-3}+ \\ (a_{n+1}+a_n a_{n+2}+a_{n+2}+a_n+3)x_{n-2})((a_{n+1}+1)x_{n-4}+ \\ (1+a_{n+1}a_{n-1}+a_{n-1})x_{n-3}+(a_n+2+a_{n+1})x_{n-2})$$

By dropping the x_{n-4} terms and using that $a_n = C$, we get

$$x_{n+2}x_{n+1} \geq ((a_{n+2}+1+a_{n+1}a_{n-1}+a_{n+2}a_{n-1}+2a_{n-1})x_{n-3}+(a_{n+1}+Ca_{n+2}+a_{n+2}+C+3)x_{n-2}) \\ ((1+a_{n+1}a_{n-1}+a_{n-1})x_{n-3}+(C+2+a_{n+1})x_{n-2})$$

By dropping the x_{n-3}^2 terms we get,

$$x_{n+2}x_{n+1} \geq a_{n-1}(Ca_{n+2}a_{n+1}+2Ca_{n+1}+2Ca_{n+2}+3C+2a_{n+2}a_{n+1}+2a_{n+1}^2+3a_{n+2}+8a_{n+1}+7) \\ +(Ca_{n+2}a_{n+1}+C^2a_{n+2}+C^2+6Ca_{n+2}+2Ca_{n+1}+7C+ \\ a_{n+1}^2+2a_{n+2}a_{n+1}+5a_{n+2}+7a_{n+1}+11)x_{n-2}x_{n-3} \quad (6)$$

Now let

$$J = a_{n-1}(Ca_{n+2}a_{n+1}+C+a_{n+2}+1)+ \\ (2Ca_{n+2}a_{n+1}+2C+2a_{n+2}a_{n+1}+3a_{n+2}+4)$$

and

$$K = (a_{n-1}(Ca_{n+2}a_{n+1}+2Ca_{n+1}+2Ca_{n+2}+3C+2a_{n+2}a_{n+1}+2a_{n+1}^2+3a_{n+2}+8a_{n+1}+7) \\ +(Ca_{n+2}a_{n+1}+C^2a_{n+2}+C^2+6Ca_{n+2}+2Ca_{n+1}+7C+ \\ a_{n+1}^2+2a_{n+2}a_{n+1}+5a_{n+2}+7a_{n+1}+11)$$

Then Equation 5 can be rewritten as

$$|r_{n+2}| \leq J\max(|r_{n-2}|, |r_{n-3}|, |r_{n-4}|)$$

and Equation 6 can be rewritten as

$$x_{n+2}x_{n+1} \geq Kx_{n-2}x_{n-3}$$

Therefore,

$$\frac{|r_{n+2}|}{x_{n+2}x_{n+1}} \leq \frac{J}{K}F_n$$

Since the derivative of $\frac{J}{K}F_n$ with respect to a_{n-1} is always negative when $a_{n-1} \geq C+1$, $a_{n+1} \geq C$ and $a_{n+2} \geq C$, then the above expression achieves its maximum in the region bounded by $a_{n-1} \geq C+1$, $a_{n+1} \geq C$ and $a_{n+2} \geq C$, when $a_{n-1} = C+1$. By making the substitution $a_{n-1} = C+1$ in the expression J we get

$$J_1 = a_{n+2}(C^2 a_{n+1} + 3Ca_{n+1} + (C + 2a_{n+1} + 4) + C^2 + 4C + 5)$$

By making the substitution $a_{n-1} = C+1$ in the expression K we get

$$\begin{aligned} K_1 = {} & a_{n+2}(C^2 a_{n+1} + 3C^2 + 4Ca_{n+1} + 11C + 4a_{n+1} + 8) + \\ & (2C^2 a_{n+1} + 4C^2 + 2Ca_{n+1}^2 + 12Ca_{n+1} + 17C + 3a_{n+1}^2 + 15a_{n+1} + 18) \end{aligned}$$

Therefore we have

$$\frac{|r_{n+2}|}{x_{n+2}x_{n+1}} \leq \frac{J_1}{K_1}F_n$$

Since the derivative of $\frac{J_1}{K_1}F_n$ with respect to a_{n+2} is always positive when $a_{n+1} \geq C$ and $a_{n+2} \geq C$, then we can replace the right hand side of the above inequality with its limit as $a_{n+2} \to \infty$. Therefore,

$$\frac{|r_{n+2}|}{x_{n+2}x_{n+1}} \leq \frac{a_{n+1}C^2 + 3a_{n+1}C + 2a_{n+1} + C + 4}{a_{n+1}C^2 + 4a_{n+1}C + 4a_{n+1} + 3C^2 + 11C + 8}$$

Since the derivative of the above expression with respect to a_{n+1} is always positive when $a_{n+1} \geq C$, then we can replace the right hand side of the above inequality with its limit as $a_{n+1} \to \infty$. Therefore,

$$\frac{|r_{n+2}|}{x_{n+2}x_{n+1}} \leq \frac{C^2 + 3C + 2}{C^2 + 4C + 4}$$

QED

Lemma 17 *If $a_n = C$, $a_{n-1} > C$, $a_{n+1} \geq C$ and $a_{n+2} \geq C$ then $\frac{|q_{n+2}|}{x_{n+2}x_{n+1}} \leq \frac{C^2+3C+2}{C^2+4C+4}G_n$*

Proof:

The proof of this lemma is identical to the proof of the previous lemma with all references to r_i replaced by references to q_i.

QED

Lemma 18 $s_0(n) = \| \left(\frac{r_n}{x_n x_{n-1}}, \frac{q_n}{x_n x_{n-1}} \right) \|$

Proof:

From Lemma 3 we know that:

$$T^{-n}_{a_1 \ldots a_n}(0,0) = \left(\frac{y_{n-1}}{x_{n-1}}, \frac{z_{n-1}}{x_{n-1}} \right)$$
$$T^{-n}_{a_1 \ldots a_n}(1,0) = \left(\frac{y_n}{x_n}, \frac{z_n}{x_n} \right)$$

Therefore,

$$\begin{aligned} s_0(n) &= \| \left(\frac{y_{n-1}}{x_{n-1}}, \frac{z_{n-1}}{x_{n-1}} \right) - \left(\frac{y_n}{x_n}, \frac{z_n}{x_n} \right) \| \\ &= \| \left(\frac{y_n x_{n-1} - x_n y_{n-1}}{x_n x_{n-1}}, \frac{z_n x_{n-1} - x_n z_{n-1}}{x_n x_{n-1}} \right) \| \\ &= \| \left(\frac{r_n}{x_n x_{n-1}}, \frac{q_n}{x_n x_{n-1}} \right) \| \end{aligned}$$

QED

Lemma 19 *If $a_n = C$, $a_{n-1} > C$, $a_{n+1} \geq C$ and $a_{n+2} \geq C$ then $s_0(i) \leq \frac{C^2+3C+2}{C^2+4C+4} \max(s_0(n-2), s_0(n-3), s_0(n-4))$ for all $i \in \{n, n+1, n+2\}$.*

Proof:

From Lemma 18 we know that,

$$\begin{aligned} \max(s_0(n-2), s_0(n-3), s_0(n-4)) = \max & \frac{|r_{n-2}|}{x_{n-2}x_{n-3}}, \frac{|r_{n-3}|}{x_{n-3}x_{n-4}}, \frac{|r_{n-4}|}{x_{n-4}x_{n-5}}, \\ & \frac{|q_{n-2}|}{x_{n-2}x_{n-3}}, \frac{|q_{n-3}|}{x_{n-3}x_{n-4}}, \frac{|q_{n-4}|}{x_{n-4}x_{n-5}}) \end{aligned}$$

and

$$s_0(i) = \max \left(\left(\frac{|r_i|}{x_i x_{i-1}}, (\frac{|q_i|}{x_i x_{i-1}} \right) \right)$$

From Lemmas 12 through 17, we know that for all $i \in \{n, n+1, n+2\}$,

$$\frac{|r_i|}{x_i x_{i-1}} \leq \frac{C^2+3C+2}{C^2+4C+4} \max(\frac{|r_{n-2}|}{x_{n-2}x_{n-3}}, \frac{|r_{n-3}|}{x_{n-3}x_{n-4}}, \frac{|r_{n-4}|}{x_{n-4}x_{n-5}})$$
$$\frac{|q_i|}{x_i x_{i-1}} \leq \frac{C^2+3C+2}{C^2+4C+4} \max(\frac{|q_{n-2}|}{x_{n-2}x_{n-3}}, \frac{|q_{n-3}|}{x_{n-3}x_{n-4}}, \frac{|q_{n-4}|}{x_{n-4}x_{n-5}})$$

Therefore, for all $i \in \{n, n+1, n+2\}$,

$$s_0(i) \leq \frac{C^2+3C+2}{C^2+4C+4} \max(s_0(n-2), s_0(n-3), s_0(n-4))$$

QED

Lemma 20 *If* $\lim_{n\to\infty} s_0(n) = 0$ *then the triangle sequence* $(a_1, a_2, a_3, \dots)$ *is unique.*

Proof:

From Lemma 18, we know that $s_0(n) = \max\left(\frac{|r_n|}{x_n x_{n-1}}, \frac{|q_n|}{x_n x_{n-1}}\right)$. Therefore,

$$s_0(n) \geq \frac{|r_n|}{x_n x_{n-1}}$$
$$s_0(n) \geq \frac{|q_n|}{x_n x_{n-1}}$$

This that $\lim_{n\to\infty} s_0(n) = 0$ implies

$$\lim_{n\to\infty} s_0(n) = 0$$
$$\lim_{n\to\infty} s_0(n) = 0$$

Hence by Lemma 7 we know that the triangle sequence is unique.

QED

Theorem 4 *If any number occurs infinitely many times in a triangle sequence, then that triangle sequence is unique.*

Proof:

Given a triangle sequence $(a_1, a_2, a_3, \dots)$, let C be the smallest term that occurs infinitely many times in the sequence.

Case 1: The sequence contains only finitely many terms a_i such that $a_i \neq C$.

Then there must exist an a_N such that for all a_i where $i > N$ the terms in the sequence are C. By some Lemma 2, i the sequence $(C, C, C, \dots)$ is unique. Therefore the triangle sequence $(a_1, a_2, a_3, \dots)$ is unique.

Case 2: The sequence contains infinitely many terms a_i such that $a_i \neq C$.

By assumption, C is the smallest term that appears infinitely often in the sequence. Therefore there are only finitely many terms less than C in the triangle sequence. Let a_m be the last term in the sequence that is less than C.

Since there are infinitely many C terms in the sequence and infinitely many terms greater than C, it is possible to select an infinite sequence of $(N_1, N_2, N_3, \dots)$

such that $N_1 > m+1$, and for all i, $a_{N_i} = C$ and $a_{N_i - 1} > C$. Note that since $N_1 > m+1$, it follows immediately that $a_{N_i+1} \geq C$ and $a_{N_i+1} \geq C$. Let

$$M = \max(s_0(N_0-2), s_0(N_0-3), s_0(N_0-4))$$

We will show that for all $n \geq N_i$,

$$s_0(n) \leq \left(\frac{C^2+3C+2}{C^2+4C+4}\right)^i M$$

First consider the base case where $i = 0$. By Lemma 11, for all $s_0(n) \geq N_0$,

$$s_0(n) \leq M$$

Now assume that for all $n \geq N_k$,

$$s_0(n) \leq \left(\frac{C^2+3C+2}{C^2+4C+4}\right)^k M$$

Therefore, we know that

$$\max(s_0(N_{k+1}-2), s_0(N_{k+1}-3), s_0(N_{k+1}-4)) \leq \left(\frac{C^2+3C+2}{C^2+4C+4}\right)^k M$$

So by Lemma 19 we know that

$$s_0(N_{k+1}) \leq \left(\frac{C^2+3C+2}{C^2+4C+4}\right)^{k+1} M$$

$$s_0(N_{k+1}+1) \leq \left(\frac{C^2+3C+2}{C^2+4C+4}\right)^{k+1} M$$

$$s_0(N_{k+1}+2) \leq \left(\frac{C^2+3C+2}{C^2+4C+4}\right)^{k+1} M$$

Hence by Lemma 11, we know that for all $n \geq N_{k+1}$,

$$s_0(n) \leq \left(\frac{C^2+3C+2}{C^2+4C+4}\right)^{k+1} M$$

Therefore, by induction we have that for all i and for all $n \geq N_i$

$$s_0(n) \leq \left(\frac{C^2+3C+2}{C^2+4C+4}\right)^i M$$

Therefore we know that

$$\lim_{n\to\infty} s_0(n) \leq \lim_{i\to\infty} \left(\frac{C^2+3C+2}{C^2+4C+4}\right)^i M$$

Therefore since $\frac{C^2+3C+2}{C^2+4C+4} < 1$,

$$\lim_{n\to\infty} s_0(n) = 0$$

Thus, by Lemma 20, the triangle sequence is unique.

QED

References

[1] Thomas Garrity. On periodic sequences for algebraic numbers. 1999.