# A Bound on the Distance from Approximation Vectors to the Plane

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**Abstract**

In this paper, we will begin by reviewing triangle sequences. One interpretation of these sequences is as a sequence of approximation vectors approaching the plane $x + \alpha y + \beta z = 0$. This paper establishes an upper bound on the distance from the $n^{th}$ approximation vector to the plane $x + \alpha y + \beta z = 0$.

## 1 Introduction

This paper will start with an overview of triangle sequences as outlined in work by Garrity [2]. We begin with a geometrical interpretation of triangle sequences. We define an iteration $T$ on the triangle:

$$\triangle = (x, y) : 1 \geq x \geq y > 0.$$

This triangle is partitioned into an infinite set of disjoint subtriangles

$$\triangle_k = (x, y) \in \triangle : 1 - x - ky \geq 0 > 1 - x - (k+1)y,$$

where $k$ is any nonnegative integer.

Define the map $T : \triangle \to \triangle \cup (x, 0) : 0 \leq x \leq 1$ by

$$T(\alpha, \beta) = \left( \frac{\beta}{\alpha}, \frac{1 - \alpha - k\beta}{\alpha} \right),$$

where $(\alpha, \beta) \in \triangle_k$.

The triangle sequence is recovered from this iteration by keeping track of the number of the triangle that the point is mapped into at each step. In other words, if $T^{k-1}(\alpha, \beta) \in \triangle_{a_k}$, then the point $(\alpha, \beta)$ will have the triangle sequence $(a_1, a_2, a_3, \dots)$.

We will recursively define a sequence of vectors as follows:
Set$C_{-2} = (1, 0, 0)$, $C_{-1} = (0, 1, 0)$, $C_0 = (0, 0, 1)$ and

$$C_n = C_{n-3} - C_{n-2} - a_n C_{n-1}$$

Let the components of $C_n$ be denoted by $C_n = (p_n, q_n, r_n)$. These vectors $C_n$ can be thought of as integer vectors approximating the plane $x + \alpha y + \beta z = 0$. We thus refer to the $C_n$ vectors as *approximation vectors.* We define positive numbers $d_n$ in the following manner:

$$d_n = (1, \alpha, \beta) \cdot C_n$$

These numbers are an indication of close the approximation vectors are to the plane $x+\alpha y+\beta z = 0$. (In fact, these numbers differ from the Euclidean distance to the plane by a constant factor). The rest of the paper concerns itself with bounding $d_n$ from above. Let $X_n = C_n \times C_{n+1}$ as defined in [1]. We shall denote the components of $X_n$ as $(x_n, y_n, z_n)$. In the next section we establish the bound of:

$$d_n < \frac{1}{x_{n+1}}$$

## 2 A Bound on $d_n$

Let $N_n$ be the matrix,

$$\begin{pmatrix} x_{n-1} & y_{n-1} & z_{n-1} \\ x_n - x_{n-1} & y_n - y_{n-1} & z_n - z_{n-1} \\ x_{n-2} & y_{n-2} & z_{n-2} \end{pmatrix}$$

We set $M_n = (C_{n-2}, C_{n-1}, C_n)$. Recall from [2] that $\det(M_n) = C_{n-2} \cdot (C_{n-1} \times C_n) = 1$.

**Lemma 1** $M_n^{-1} = N_n$

*Proof:* We know that

$$N_n M_n = \begin{pmatrix} x_{n-1} & y_{n-1} & z_{n-1} \\ x_n - x_{n-1} & y_n - y_{n-1} & z_n - z_{n-1} \\ x_{n-2} & y_{n-2} & z_{n-2} \end{pmatrix} \begin{pmatrix} p_{n-2} & p_{n-1} & p_n \\ q_{n-2} & q_{n-1} & q_n \\ r_{n-2} & r_{n-1} & r_n \end{pmatrix}$$

By performing the above multiplication, we get

$$N_n M_n = \begin{pmatrix} C_{n-2} \cdot X_{n-1} & C_{n-1} \cdot X_{n-1} & C_n \cdot X_{n-1} \\ C_{n-2} \cdot X_n - C_{n-2} \cdot X_{n-1} & C_{n-1} \cdot X_n - C_{n-1} \cdot X_{n-1} & C_n \cdot X_n - C_n \cdot X_{n-1} \\ C_{n-2} \cdot X_{n-2} & C_{n-1} \cdot X_{n-2} & C_n \cdot X_{n-2} \end{pmatrix}$$

Since for all $k$, we know that $X_k = C_k \times C_{k+1}$, then

$$C_k \cdot X_k = 0$$
$$C_k \cdot X_{k-1} = 0$$
$$C_k \cdot X_{k+1} = 1$$

Additionally,

$$
\begin{aligned}
&C_{n-2} \cdot X_n = C_{n-2} \cdot (X_{n-3} + a_n X_{n-2} + X_{n-1}) \\
&C_{n-2} \cdot X_n = C_{n-2} \cdot X_{n-3} + a_n C_{n-2} \cdot X_{n-2} + C_{n-2} \cdot X_{n-1} \\
&C_{n-2} \cdot X_n = 1
\end{aligned}
$$

and

$$
\begin{aligned}
&C_n \cdot X_{n-2} = (C_{n-3} - C_{n-2} - a_n C_{n-1}) \cdot X_{n-2} \\
&C_n \cdot X_{n-2} = C_{n-3} \cdot X_{n-2} - C_{n-2} \cdot X_{n-2} + a_n C_{n-1} \cdot X_{n-2} \\
&C_n \cdot X_{n-2} = 1
\end{aligned}
$$

Therefore,

$$
N_n M_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Hence, $N_n = M_n^{-1}$

□

**Lemma 2** *The following three equalities are true:*

$$
\begin{aligned}
1 &= d_n x_{n-2} + d_{n-2} x_{n-1} + d_{n-1} x_n - d_{n-1} x_{n-1} \\
\alpha &= d_n y_{n-2} + d_{n-2} y_{n-1} + d_{n-1} y_n - d_{n-1} y_{n-1} \\
\beta &= d_n z_{n-2} + d_{n-2} z_{n-1} + d_{n-1} z_n - d_{n-1} z_{n-1}
\end{aligned}
$$

*Proof:* We know that

$$
(1, \alpha, \beta) M_n = (d_{n-2}, d_{n-1}, d_n)
$$

Therefore, since $M_n^{-1} = N_n$,

$$
(1, \alpha, \beta) = (d_{n-2}, d_{n-1}, d_n) N_n
$$

Therefore,

$$
\begin{aligned}
1 &= d_n x_{n-2} + d_{n-2} x_{n-1} + d_{n-1} x_n - d_{n-1} x_{n-1} \\
\alpha &= d_n y_{n-2} + d_{n-2} y_{n-1} + d_{n-1} y_n - d_{n-1} y_{n-1} \\
\beta &= d_n z_{n-2} + d_{n-2} z_{n-1} + d_{n-1} z_n - d_{n-1} z_{n-1}
\end{aligned}
$$

□

**Lemma 3**

$$
\begin{aligned}
\alpha &= \frac{y_{n-1} + \beta_n y_{n-2} + \alpha_n y_n - \alpha_n y_{n-1}}{x_{n-1} + \beta_n x_{n-2} + \alpha_n x_n - \alpha_n x_{n-1}} \\
\beta &= \frac{z_{n-1} + \beta_n z_{n-2} + \alpha_n z_n - \alpha_n z_{n-1}}{x_{n-1} + \beta_n x_{n-2} + \alpha_n x_n - \alpha_n x_{n-1}}
\end{aligned}
$$

*Proof:* From Lemma 2, we know that

$$1 = d_{n-2}x_{n-1} + d_n x_{n-2} + d_{n-1}x_n - d_{n-1}x_{n-1}$$

By recalling that $\alpha_n = \frac{d_{n-1}}{d_{n-2}}$ and $\beta_n = \frac{d_n}{d_{n-2}}$, we can divide through by $d_{n-2}$ to get

$$\frac{1}{d_{n-2}} = x_{n-1} + \beta_n x_{n-2} + \alpha_n x_n - \alpha_n x_{n-1}$$

Or equivalently,

$$d_{n-2} = \frac{1}{x_{n-1} + \beta_n x_{n-2} + \alpha_n x_n - \alpha_n x_{n-1}} \tag{1}$$

Also from Lemma 2, we have that

$$\alpha = d_{n-2}y_{n-1} + d_n y_{n-2} + d_{n-1}y_n - d_{n-1}y_{n-1}$$
$$\beta = d_{n-2}z_{n-1} + d_n z_{n-2} + d_{n-1}z_n - d_{n-1}z_{n-1}$$

Factoring out a $d_{n-2}$ from the right hand sides yields

$$\alpha = d_{n-2}(y_{n-1} + \beta_n y_{n-2} + \alpha_n y_n - \alpha_n y_{n-1})$$
$$\beta = d_{n-2}(z_{n-1} + \beta_n z_{n-2} + \alpha_n z_n - \alpha_n z_{n-1})$$

Substituting in for $d_{n-2}$ from Equation (1), we get

$$\alpha = \frac{y_{n-1} + \beta_n y_{n-2} + \alpha_n y_n - \alpha_n y_{n-1}}{x_{n-1} + \beta_n x_{n-2} + \alpha_n x_n - \alpha_n x_{n-1}}$$
$$\beta = \frac{z_{n-1} + \beta_n z_{n-2} + \alpha_n z_n - \alpha_n z_{n-1}}{x_{n-1} + \beta_n x_{n-2} + \alpha_n x_n - \alpha_n x_{n-1}}$$

□

**Theorem 1** $d_n < \frac{1}{x_{n+1}}$

*Proof:* From Lemma 2, we know that

$$d_{n+1}x_{n-1} + d_{n-1}x_n + d_n x_{n+1} - d_n x_n = 1$$

Since $d_{n+1} > 0$ and $x_{n-1} > 0$, we can write the inequality:

$$d_{n-1}x_n + d_n x_{n+1} - d_n x_n < 1$$

Since the $d_n$ terms are decreasing, we can substitute $d_n$ for $d_{n-1}$ and maintain the inequality:

$$d_n x_n + d_n x_{n+1} - d_n x_n < 1$$

This implies that

$$d_n < \frac{1}{x_{n+1}}$$

□

## References


[1] T. Cheslack-Postava, A. Diesl, M. Lepinski, and A. Schuyler. Some results concerning uniqueness of triangle sequences. 1999.

[2] Thomas Garrity. On periodic sequences for algebraic numbers. 1999.