# FIRST EXAM–SOLUTIONS

MATH 211, FALL 2006, WILLIAMS COLLEGE

These are the instructor's solutions to the first midterm exam.

## 1. Problem One

Compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 5 \\ 2 & 0 & -1 & 0 \\ 0 & 7 & 0 & 1 \end{pmatrix}.$$

1.1. **Solution.** The answer is $\det(A) = \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 5 \\ 2 & 0 & -1 & 0 \\ 0 & 7 & 0 & 1 \end{vmatrix} = 160.$

## 2. Problem Two

Consider the following system of linear equations.

$$\begin{array}{cccccc} 4x & +3y & +z & = & 8 \\ 2x & +y & -z & = & 2 \\ x & -y & +z & = & 1 \end{array}$$

- Write the matrix form of this system.
- Find the set of solutions to the system of linear equations by any method involving matrices.

2.1. **Solution.** The matrix form of the equation is

$$\begin{pmatrix} 4 & 3 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 1 \end{pmatrix}.$$

The system has a unique solution: the vector $(1, 1, 1)^t$, that is, $x = 1, y = 1, z = 1$.

## 3. Problem Three

- Add the matrices $C = \begin{pmatrix} 0 & -5 \\ 3 & 7 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 1 \\ -3 & 6 \end{pmatrix}$.
- Compute the inner product of the vectors $v_1 = \begin{pmatrix} 3 & 4 \end{pmatrix}^t$ and $v_2 = \begin{pmatrix} -1 & 2 \end{pmatrix}^t$
- Find the inverse of the matrix $E = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 5 \\ 0 & -1 & 1 \end{pmatrix}$, if it exists.

3.1. **Solution.** We find that $C + D = \begin{pmatrix} 1 & -4 \\ 0 & 13 \end{pmatrix}$, and also that $\langle v, w \rangle = 5$. The inverse of $E$ is the matrix

$$E^{-1} = \begin{pmatrix} -5 & 2 & 10 \\ 3 & -1 & 5 \\ 3 & -1 & 6 \end{pmatrix}.$$

## 4. Problem Four

Suppose that $A$ is a square matrix. Show that $\text{null}(A) = \{0\}$ if and only if the columns of $A$ are linearly independent.

4.1. **Solution.** We write the matrix $A$ as a collection of column vectors $v_1, \ldots, v_n$, that is,

$$A = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \end{pmatrix}.$$

Then for a vector $x = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}^t$ we see that matrix multiplication $Ax$ is the same as a linear combination of the vectors $v_i$ with weights $x_i$.

$$Ax = x_1 v_1 + \ldots + x_n v_n. \tag{1}$$

So we see that $x$ is a solution to $Ax = 0$ if and only if $x_1 v_1 + \ldots + x_n v_n = 0$.

If $A$ has a nontrivial null space, there is a non-zero vector $x$ such that (1) is zero. But this means that the linear combination in (1) can be made zero by a set of coefficients which are not all zero, and hence the column vectors are linearly dependent.

If the null space is trivial, the only way that (1) is zero is if $x = 0$. That is, the only way to make zero as a linear combination of the column vectors is with all coefficients zero. Hence the vectors are linearly independent.

## 5. Problem Five

Write down a system of two equations in three unknowns that fails to have a solution. Draw a picture that explains why your system fails to have a solution and explain the relevance of your picture.

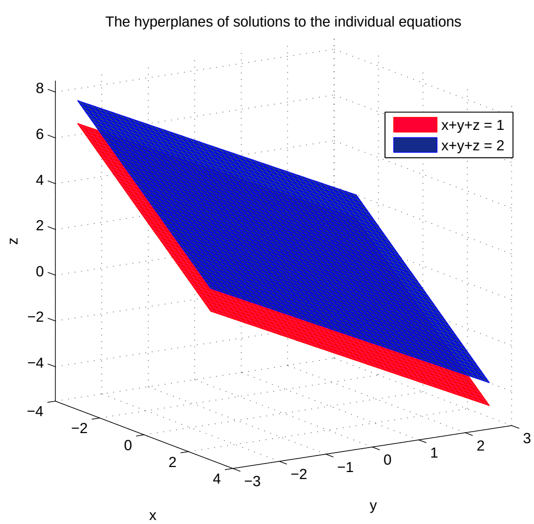
### 5.1. **Solution.** My solution is the system of equations

$$\begin{cases} x & +y & +z & = & 1 \\ x & +y & +z & = & 2 \end{cases}$$

*First explanation:*

The set of solutions to each equation is a hyperplane (i.e. a regular plane) in $\mathbb{R}^3$. So, the set of solutions to the system is the intersection of these hyperplanes.

However, these hyperplanes are parallel but distinct, and therefore never intersect, so there is no solution to the system. The picture looks like this one below.



*Second explanation:*

The system is equivalent to the matrix equation

$$(x+y+z)\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}.$$

However, the vector $\begin{pmatrix}1 & 2\end{pmatrix}$ does not lie in the line formed by scalar multiples of the vector $\begin{pmatrix}1 & 1\end{pmatrix}^t$, as can be seen by plotting them in the plane. (This line is the column space of the matrix in the corresponding matrix equation $\begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$.) Therefore, the system can not have a solution. The corresponding picture is below.