HOMEWORK ASSIGNMENT # 1

MATH 211, FALL 2006, WILLIAMS COLLEGE

Abstract. These are the instructor's solutions to the first homework.

1. Problem One

Describe, in your own words, the geometric way to multiply complex numbers. Draw pictures and explain in full sentences. Integrate any formulae you use into your text. [*Hint: Try to write a small section of textbook.*]

1.1. **Solution.** We discussed this in class. One begins with the polar coordinate representations of two complex numbers

$$z = r(\cos\theta + i\sin\theta) \quad \text{and} \quad w = r'(\cos\theta' + i\sin\theta').$$

Where r and r' represent the distance from the origin, and θ and θ' represent the counterclockwise angle from the positive real axis.

By DeMoivre's theorem, we may write these as $z = re^{i\theta}, w = r'e^{i\theta'}$ and multiply to get $zw = rr'e^{i(\theta+\theta')}$. This means that zw has distance to the origin equal to the product of the distances for z and w, and angle equal to the sum of the angles for z and w.

If we think of this as multiplying the vector z by the scalar w, the effect is to combine two geometric operations. Scale the vector by the size of w and then rotate it counterclockwise by an angle equal to that of w.

I haven't found a good way to put the pictures into this document.

2. Problem Two

Recall that $\mathbb{R}^n \subset \mathbb{C}^n$ in a natural way. But there is also a way to think of $\mathbb{C}^n$ as a real vector space. Thinking of $\mathbb{C}$ as ordered pairs of real numbers, we can forget the complex multiplication and identify the complex n-vector $v = (a_1 + ib_1, \ldots, a_n + ib_n)$ with the real $2n$-vector $v' = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$. This gives a view of $\mathbb{C}^n$ as a copy of $\mathbb{R}^{2n}$. How are the notions of length in these two spaces related? That is, how is $||v||_{\mathbb{C}^n}$ related to $||v'||_{\mathbb{R}^{2n}}$?

Date: September 12, 2006.

2.1. **Solution.** The two notions of length agree. I'll sketch how to see this.

Let v, v' be as stated in the problem. Then by definition
$$||v||_{\mathbb{C}^n} = \sqrt{\langle v, v\rangle} = \sqrt{(a_1 + ib_1)\cdot(a_1 - ib_1) + \cdots + (a_n + ib_n)\cdot(a_n - ib_n)}\,.$$
(Note that I've already performed the conjugation from the Hermitian product.) Doing the complex multiplication, or noting that $z \cdot \overline{z} = |z| = a^2 + b^2$ for a complex number $z = a + ib$, we see that
$$||v||_{\mathbb{C}^n} = \sqrt{a_1^2 + b_1^2 + \cdots + a_n^2 + b_n^2}\,.$$

And it is not too hard to see that, by definition,
$$||v'||_{\mathbb{R}^{2n}} = \sqrt{\langle v', v'\rangle} = \sqrt{a_1^2 + b_1^2 + \cdots + a_n^2 + b_n^2}\,.$$

3. Problem Three

Describe the differences between the algebra of matrices and the algebra of real (or complex) numbers. Highlight your discussion with examples.

3.1. **Solution.** The structure of addition is basically the same. Addition is commutative, associative, and there are always additive inverses (negatives). The differences come from the multiplication. Multiplication is still associative, and there is a distributive law (for both right and left multiplication). There is still a "multiplicative unit": the identity matrix I has $AI = IA = A$ for every matrix A. However, multiplication is no longer commutative in general. For example, consider the matrices
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
It is not hard to compute that $AB = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ is not the same thing as $BA = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. The other main difference is that we may not always "divide". For numbers, we can always divide by a non-zero element. That is, every non-zero number a has a multiplicative inverse a^{-1} such that $aa^{-1} = 1$. Consider the non-zero matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. There is no matrix B such that $AB = BA = I$. To see this, suppose B exists and has the form $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that the lower left entry of both AB and BA is always 0, no matter what a, b, c, d are, so it is impossible to get $AB = I$ or $BA = I$.

4. Problem Four

List two linear algebra textbooks (other than our text) that can be found in either Schow or the Math commons. Give title, author and title of the first chapter.

4.1. **Solution.** There are a great many possible answers here.

5. Problem Five

Is it true that every real square matrix can be written as the sum of symmetric matrix and a skew symmetric matrix? If so, show how to do it. If false, give a counter example.

5.1. **Solution.** It is true. Given an arbitrary square matrix A consider the matrices $B = \frac{1}{2}(A + A^t)$ and $C = \frac{1}{2}(A - A^t)$, where A^t denotes the transpose of A. It is clear that $A = B + C$. Also, note that the transpose of B is

$$B^t = \frac{1}{2}(A + A^t)^t = \frac{1}{2}(A^t + (A^t)^t).$$

Since $(A^t)^t = A$, we see that $B^t = B$. Thus B is symmetric. A similar check shows that $C^t = -C$, so that C is skew-symmetric.

6. Problem Six

(1) Suppose that L_1, L_2 are two hyperplanes in $\mathbb{R}^2$. What shape can their intesection take? Give some representative examples of the possibilities.
(2) Do the same for two hyperplanes L_1, L_2 in $\mathbb{R}^3$.
(3) Do the same for three hyperplanes L_1, L_2, L_3 in $\mathbb{R}^3$.

6.1. **Solution.** We'll tackle the three parts one at a time. It is really helpful to draw pictures for these.

(1) A hyperplane in $\mathbb{R}^2$ is just a line, so the question asks for the possible intersections of two lines. The possibilities are:
   - **A single point:** This happens when the lines have different slope. For example, $2x + y = 1$ and $3x + y = 1$ intersect at exactly the point $x = 0, y = 1$.
   - **A line:** This happens when the two lines coincide. For example take L_1 to be $x + y = 2$, L_2 to be $2x + 2y = 4$, and their intersection is still all of the line $x + y = 2$.
   - **Empty:** This happens when the lines are parallel but distinct. For example, $x + y = 0$ and $x + y = 2$ never touch.

(2) A hyperplane in $\mathbb{R}^3$ is a two dimensional plane. The possibilities are:

A plane: If the two planes are equal, then we keep all the points. For example, take $L_1 = L_2 = \{x + y + z = 0\}$.

A line: If the two planes are not parallel, we get a single line. For example, the planes $x + y + z = 0$ and $2x + y + 2z = 0$ intersect in the line described by $\{y = 0, z = -x\}$.

empty: This occurs when the planes are parallel, but distinct. For example, $x + y + z = 0$ and $x + y + z = 24$ do not touch.

(3) For three planes in $\mathbb{R}^3$ we have a few more possibilities.

Empty: This happens often, as even if each pair of planes intersects, the common intersection may be empty. For example consider the planes $x = 0$, $z = 0$ and $x + z = 1$

A point: This happens when the pairs of planes intersect in lines, and the lines intersect in a point. For example, take $x = 0$, $y = 0$ and $x + y + z = 1$.

A line: This can happen even if all three planes are distinct. For example, take the planes $2x + y = 0$, $x + y = 0$ and $y - 2x = 0$. These intersect in the "vertical" line $\{x = 0, y = 0\}$.

A plane: This happens when all three planes are the same geometric object. Like $x+2y-z = 3$, $16x+32y-16y = 48$, and $-5x - 10y + 5z = -15$.