MATH 211 HOMEWORK 10 SOLUTIONS

① Recall that $\det(X) = \det(X^t)$ for any $n \times n$ matrix X. So we see that

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff \det(A - \lambda I) = 0 \\ &\iff \det((A-\lambda I)^t) = 0 \\ &\iff \det(A^t - \lambda I) = 0 \\ &\iff \lambda \text{ is an eigenvalue of } A^t \end{aligned}$$

② $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$,
$$\begin{aligned} p_A(t) = \det(A - tI) &= \det\begin{pmatrix} 3-t & 1 \\ 5 & 2-t \end{pmatrix} \\ &= (3-t)(2-t) - 5 \\ &= 6 - 5t + t^2 - 5 = t^2 - 5t + 1 \end{aligned}$$

evals/roots ?

$$\begin{aligned} 0 = p_A(t) = t^2 - 5t + 1 &= t^2 - 5t + \tfrac{25}{4} - \tfrac{21}{4} \\ &= (t - 5/2)^2 - \tfrac{21}{4} \end{aligned}$$

so $\cancel{t}$ $\lambda_{1,2} = \frac{5}{2} \pm \frac{\sqrt{21}}{2}$

$\lambda_1 = \frac{5 + \sqrt{21}}{2}$ find eigenspace $E_{\lambda_1} = \text{null}(A - \lambda_1 I)$

$$A - \lambda_1 I = \begin{pmatrix} 3-\lambda_1 & 1 \\ 5 & 2-\lambda_1 \end{pmatrix} \sim \begin{pmatrix} 3-\lambda_1 & 1 \\ 0 & ? \end{pmatrix}$$

$$2 - \lambda_1 - \frac{5}{3-\lambda_1}(1) = 0$$

$$\Rightarrow E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} 1 \\ \lambda_1 - 3 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ \frac{-1+\sqrt{21}}{2} \end{pmatrix} \right\}$$

Similarly $E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} 1 \\ \frac{-1-\sqrt{21}}{2} \end{pmatrix} \right\}$

② (cont.)

So we must have

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = A = P^{-1}DP$$
$$= \begin{pmatrix} 1 & 1 \\ \frac{-1+\sqrt{21}}{2} & \frac{-1-\sqrt{21}}{2} \end{pmatrix}^{-1} \begin{pmatrix} \frac{5+\sqrt{21}}{2} & 0 \\ 0 & \frac{5-\sqrt{21}}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{-1+\sqrt{21}}{2} & \frac{-1-\sqrt{21}}{2} \end{pmatrix}$$

$B = \begin{pmatrix} 8 & 1 \\ -1 & 6 \end{pmatrix}$. $P_B(t) = \det \begin{pmatrix} 8-t & 1 \\ -1 & 6-t \end{pmatrix}$

$$= (8-t)(6-t) + 1$$
$$= 48 - 14t + t^2 + 1$$
$$= t^2 - 14t + 49 = (t-7)^2.$$

There is only one eigenvalue, $\lambda = 7$.

$$E_\lambda = \text{null}(B - 7I) = \text{null}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \text{null}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$= \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ this is one dimensional.

Thus it is impossible to diagonalize B, as one cannot pick a basis of $\mathbb{R}^2$ consisting of eigenvectors.

③ $A = \begin{pmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix}$ This is symmetric. So by spectral theorem we can choose P orthogonal.

$p_A(t) = \det(A - tI) = -t^3 + 6t^2 + 135t + 400.$

A root of this must divide $400 = 2^4 \cdot 5^2$ (if an integer!)
So we check (hoping...) $2, -2, 4, -4, 5, -5$
Bingo! $p_A(-5) = 0$ so divide!

$$\begin{array}{r|l} & -t^2 + 11t + 80 \\ t+5 & -t^3 + 6t^2 + 135t + 400 \\ & -t^3 - 5t^2 \\ \hline & 11t^2 \\ & 11t^2 + 55t \\ \hline & 80t \\ & 80t + 400 \\ \hline & 0 \end{array}$$

$\lambda_1 = -5$

and the roots of $-t^2 + 11t + 80 = q(t)$ are

$$\lambda_{2,3} = \frac{-11 \pm \sqrt{121 - 4(80)(-1)}}{-2} = \frac{+11 \mp \sqrt{441}}{2} =$$

$$= \frac{11 \mp 21}{2} = -5, 16.$$

So our eigenvalues are -5 (twice) and 16.

③ (Cont.)

Now we find bases of the eigenspaces:

$$E_{-5} = \text{null}\left(A+5I\right) = \text{null}\begin{pmatrix} 16 & -8 & 4 \\ -8 & -4 & -2 \\ 4 & -2 & 1 \end{pmatrix}$$

$$= \text{null}\begin{pmatrix} 4 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right\}$$

We apply Gramm-Schmidt to obtain an orthogonal basis

$$v_1 = (1 \quad 0 \quad -4)^t,$$
$$v_2 = (1 \quad 2 \quad 0)^t - \frac{1}{17}(1 \quad 0 \quad -4)^t = \left(\tfrac{16}{17} \quad 2 \quad \tfrac{4}{17}\right)^t$$

and we normalize these to get an orthonormal basis for E_{-5}

$$u_1 = \begin{pmatrix} 1/\sqrt{17} \\ 0 \\ -4/\sqrt{17} \end{pmatrix}, \quad u_2 = \begin{pmatrix} 16/\sqrt{1428} \\ 34/\sqrt{1428} \\ 4/\sqrt{1428} \end{pmatrix}$$

$$E_{16} = \text{null}\left(A-16I\right) = \text{null}\begin{pmatrix} -5 & -8 & 4 \\ -8 & -17 & -2 \\ 4 & -2 & -20 \end{pmatrix}$$

$$= \text{null}\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}\right\}$$

We normalize this to get $u_3 = \begin{pmatrix} 4/\sqrt{21} \\ -2/\sqrt{21} \\ 1/\sqrt{21} \end{pmatrix}$

③ (cont.)

Therefore,

$$A = \begin{pmatrix} 1/\sqrt{17} & 16/\sqrt{1428} & 4/\sqrt{21} \\ 0 & 34/\sqrt{1428} & -2/\sqrt{21} \\ -4/\sqrt{17} & 4/\sqrt{1428} & 1/\sqrt{21} \end{pmatrix}^{-1} \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 16 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{17}} & \frac{16}{\sqrt{1428}} & \frac{4}{\sqrt{21}} \\ 0 & \frac{34}{\sqrt{1428}} & \frac{-2}{\sqrt{21}} \\ \frac{-4}{\sqrt{17}} & \frac{4}{\sqrt{1428}} & \frac{1}{\sqrt{21}} \end{pmatrix}$$

is an orthogonal similarity with a diagonal matrix.

~~NEXT~~ MATRIX!

$$B = \begin{pmatrix} 1 & 3 & 1 \\ 3 & -7 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

$$P_B(t) = \det(B - tI) = -t^3 - 5t^2 + 23t$$

$$= -(t-0)(t^2 + 5t - 23)$$

$$\lambda_1 = 0 \quad , \quad \lambda_2, \lambda_3 ?$$

$$\lambda_{2,3} = \frac{-5 \pm \sqrt{25 - 4(23)}}{2} = \frac{-5 \pm \sqrt{117}}{2}$$

Now find eigenspaces:

$$E_0 = \text{null}(B - 0I) = \text{null}(B) = \text{null}\begin{pmatrix} 1 & 3 & 1 \\ 3 & -7 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

$$= \text{null}\begin{pmatrix} 1 & 0 & 7/16 \\ 0 & 1 & 3/16 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} -7/16 \\ -3/16 \\ 1 \end{pmatrix} \right\}$$

$$v_1 = \begin{pmatrix} -7 \\ -3 \\ 16 \end{pmatrix} \quad \text{is nice.}$$

③ (cont.)

let $\lambda_2 = \dfrac{-5+\sqrt{117}}{2}$

I'll do the formal row reduction with λ_2 b/c then the same steps are valid when you use λ_3!

$$E_{\lambda_2} = \text{null}\begin{pmatrix} 1-\lambda_2 & 3 & 1 \\ 3 & -7-\lambda_2 & 0 \\ 1 & 3 & 1-\lambda_2 \end{pmatrix}$$

$$= \text{null}\begin{pmatrix} 1-\lambda_2 & 3 & 1 \\ 3 & -7-\lambda_2 & 0 \\ 1 & 3 & 1-\lambda_2 \end{pmatrix} ???$$

ROW REDUCE! $\lambda = \lambda_2$

$$\sim \begin{pmatrix} 1 & 3 & 1-\lambda \\ 3 & -7-\lambda & 0 \\ 1-\lambda & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1-\lambda \\ 0 & -16-\lambda & -3+3\lambda \\ 0 & 3\lambda & 2\lambda-\lambda^2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 1-\lambda \\ 0 & -16-\lambda & -3+3\lambda \\ 0 & 1 & \frac{2}{3}-\frac{\lambda}{3} \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1-\lambda \\ 0 & 1 & \frac{2}{3}-\frac{1}{3}\lambda \\ 0 & 0 & 23-5\lambda-\lambda^2 \end{pmatrix}$$

$= 0$ b/c λ_2 a root of this!

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{2}{3}-\frac{\lambda}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

so $E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 3 \\ \lambda_2-2 \\ 3 \end{pmatrix}\right\}$, $v_2 = \begin{pmatrix} 3 \\ \frac{-9+\sqrt{117}}{2} \\ 3 \end{pmatrix}$

and by the same work $E_{\lambda_3} = \text{span}\left\{\begin{pmatrix} 3 \\ \lambda_3-2 \\ 3 \end{pmatrix}\right\}$

$v_3 = \begin{pmatrix} 3 \\ \frac{-9-\sqrt{117}}{2} \\ 3 \end{pmatrix}$