HOMEWORK ASSIGNMENT SOLUTIONS# 2

MATH 211, FALL 2006, WILLIAMS COLLEGE

Abstract. These are the instructors solutions for assignment 2.

1. Problem One

Use the elimination/back-solving algorithm to solve the following systems. (i.e. put the system in row echelon form and use back-solving.)

(1)

$$\begin{cases} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = & 7 \\ 3x_1 & +2x_2 & +x_3 & +x_4 & -3x_5 & = & -2 \\ & +x_2 & +2x_3 & +2x_4 & +6x_5 & = & 23 \\ 5x_1 & +4x_2 & +3x_3 & +3x_4 & -x_5 & = & 12 \end{cases}$$

(2)

$$\begin{cases} -2x_1 & +x_2 & +6x_3 & = & 18 \\ 5x_1 & & +8x_3 & = & -16 \\ 3x_1 & +2x_2 & -10x_3 & = & -3 \end{cases}$$

1.1. **Solution.** The exact form of the work to a solution depends a lot upon your choices, so I'll omit this. However, one should put the systems into an echelon/triangular form and then use back-solving.

(1) This system has lots of redundancy. It turns out that there are really only two equations to keep in the end. (Eliminate the x_1 terms from the second and fourth equations to see this.) I got a solution, with x_3, x_4 and x_5 free variables, of

$$\{(-16 + x_3 + x_4 + 5x_5, 23 - 2x_2 - 2x_3 - 6x_5, x_3, x_4, x_5)\}.$$

(2) The answer is $x_1 = -4, x_2 = 7, x_3 = 1/2$. As a guide to how I think about these, I'll write down the operations I took. You should try to follow them.
Switch rows 1 and 2. Add $2/5$ of equation 1 to equation 2. Add $-3/5$ equation 1 to equation 3. Add -2 equation 2 to equation 3. Now the system is triangular.

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2. Problem Two

(1) Consider the hyperplane $\mathcal{H} = \{a_1x_1 + \ldots + a_nx_n = b\}$. Show that the vector $n = (a_1, \ldots, a_n)$ is normal to $\mathcal{H}$.
(2) Show that two planes are parallel if and only if their normal vectors are proportional.

2.1. **Solution.** Suppose that $P = (p_1, \ldots, p_n)$ and $P' = (p'_1, \ldots, p'_n)$ are points in $\mathcal{H}$. Then the vector $P - P'$ is parallel to $\mathcal{H}$. Similarly, any vector parallel to $\mathcal{H}$ can be written as the difference of two points in $\mathcal{H}$.

By virtue of lying in $\mathcal{H}$, the coordinates for our points must satisfy the equation defining $\mathcal{H}$. So, subtracting these equations, we see that the coordinates of $P - P' = (p_1 - p'_1, \ldots, p_n - p'_n)$ satisfy

$$\langle n, P - P' \rangle = a_1(p_1 - p'_1) + \ldots + a_n(p_n - p'_n) = 0.$$

Thus, n is orthogonal to $P - P'$ since this covers all cases of vectors parallel to $\mathcal{H}$, n must be orthogonal to $\mathcal{H}$.

3. Problem Three

Use Gaussian elimination (i.e. go to reduced row echelon form) to find the solution set to the following systems.

(1)
$$\begin{cases} x_1 & +x_2 & = & 4 \\ 2x_1 & -3x_2 & = & 7 \\ 3x_1 & -2x_2 & = & 11 \end{cases}$$

(2)
$$\begin{cases} x_1 & +x_2 & = & 4 \\ 2x_1 & -3x_2 & = & 7 \\ 3x_1 & +2x_2 & = & 8 \end{cases}$$

(3)
$$\begin{cases} 3x_1 & +x_2 & -x_3 & +2x_4 & = & 1 \\ 2x_1 & -x_2 & +x_3 & +x_4 & = & 4 \\ 8x_1 & +x_2 & +x_3 & +5x_4 & = & 6 \end{cases}$$

3.1. **Solution.** Again, one should endeavor to put the systems into reduced row echelon form. I'll leave out the mess of calculation.

System one looks overdetermined, but is consistent (equation 3 is the sum of equations 1 and 2). The solution is $x_1 = 19/5, x_2 = 1/5$.

System two is inconsistent. It has no solution.

System three has a free variable of x_4. Here's what I did: Add $-3/2$ equation 1 to equation 2. Add $-8/3$ equation 1 to equation 3. Add -1 equation 2 to equation 3. At this point, the third equation is pretty simple: $2x_3 = 0$. (Yay!) Multiply equation 3 by $1/2$. Add $-5/3$ equation 3 to equation 2, and add equation 3 to equation 1. Finally, add $3/5$ equation 2 to equation 1. I also cleared the constants a bit by multiplying equation 1 by $1/3$ and equation 2 by $-3/5$. This puts the system into reduced row echelon form.

Then one can read off the solution set to be (in parametric form)

$$\{(1-(3/5)t, -2-(1/5)t, 0, t)\}.$$

4. Problem Four

Use the augmented matrix method to

(1) solve the system

$$\begin{cases} x_1 & +2x_2 & +4x_3 & = & 1 \\ 2x_1 & +x_2 & +5x_3 & = & 0 \\ 3x_1 & -x_2 & +6x_3 & = & 1 \end{cases}$$

(2) find the inverse to the matrix

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \\ 1 & 0 & 0 \end{pmatrix}.$$

Show both of your answers are correct using matrix mulitiplication.

4.1. Solution.

The solution to the first part is $x_1 = -17/3, x_2 = -2, x_3 = 8/3$. This can be checked by performing the multiplication:

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 3 & -1 & 6 \end{pmatrix} \begin{pmatrix} -17/3 \\ -2 \\ 8/3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The second part can be done similarly, and the answer is

$$A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & -3 \\ -5 & 3 & 7 \end{pmatrix}.$$

5. Problem Five

What is the "expected dimension" of the solution set to a system of n linear equations in m unknowns? That is, what happens most often? Be sure to discuss the possible cases of $m < n$, $m = n$, and $m > n$. (I don't expect a proof, but I want some discussion as to why that is the expected answer.)

5.1. **Solution.** Think of the system as evolving. First we look for solutions to the first equation, and then look for simultaneous solutions to the first two. Then we look for simultaneous solutions to the first three equations, and so on.

The solution set to one equation is a hyperplane, and it has dimension one less than the given number of variables for the whole system. Each time one intersects with a new hyperplane, the dimension goes down by one. So, each time we add an equation we cut down the dimension of the solution set by 1.

This assumes that nothing funny happens. Funny stuff includes: adding a new equation that is a linear combination of the previous equations (i.e. adding a hyperplane that doesn't change the intersection– think of three lines through a point), adding an equation that make the system degenerate (adding a hyperplane that is parallel but not equal to one of the previous ones).

So, in the general case we expect an $m - n$ dimensional solution set. If $m > n$ then this is fine. If $m = n$, this tell us to expect a unique solution. If $n > m$ we should expect no solutions.

But remember, this only works if nothing funny happens. You always have to check.