HOMEWORK ASSIGNMENT # 3

MATH 211, FALL 2006, WILLIAMS COLLEGE

Abstract. These are the instructor's solutions.

1. Problem: On LU decompositions

This problem will lead you through understanding the LU decomposition of a matrix. Your answer for this problem should show all of the work indicated in detail. Consider the matrix

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix}.$$

- Working left to right and top to bottom, apply multiplications by elementary matrices to put the matrix into row echelon form, U. Keep careful track of the matrices you use, and the order you apply them.
- Multiply the elementary matrices you used in the last step in the correct order to write a matrix E, and write out the matrix equation $EA = U$.
- In each elimination step from the first part of the problem, the number $m_{ji} = -a_{ji}/a_{ii}$ you used to scale the coefficients is called a *multiplier*. Write down the matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & -m_{32} & 1 \end{pmatrix}.$$

- Show that $E^{-1} = L$.
- Conclude that $A = L \cdot U$. Check this by performing the matrix multiplication.
- Use the LU decomposition you just found to solve the system $Ax = b$ for $b = \begin{pmatrix} 1 & 1 & 2 \end{pmatrix}^t$ by the "two backsolvings" method we discussed in class.

Note: not every matrix has an LU decomposition. If one must switch rows when doing the forward elimination, then on must keep track of these. This results in a PLU decomposition, where the P is a permutation matrix, which keeps track of the row switching.

1.1. **solution.** One adds (-1/2)row 1 to row 2, adds (-2)row 1 to row 3 and then (3)row 2 to row 3. So the multipliers are $m_{21} = -1/2$, $m_{31} = -2$ and $m_{32} = 3$, and the matrix equation we have found is just

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\-2 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\-1/2 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}2 & 4 & 2\\1 & 5 & 2\\4 & -1 & 9\end{pmatrix} = \begin{pmatrix}2 & 4 & 2\\0 & 3 & 1\\0 & 0 & 8\end{pmatrix}.$$

This is just $EA = U$ for

$$E = \begin{pmatrix}1 & 0 & 0\\-1/2 & 1 & 0\\-3/2 & 3 & 1\end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix}2 & 4 & 2\\0 & 3 & 1\\0 & 0 & 8\end{pmatrix}.$$

It is then not difficult to check that

$$E^{-1} = L = \begin{pmatrix}1 & 0 & 0\\1/2 & 1 & 0\\2 & -3 & 1\end{pmatrix},$$

and that

$$A = \begin{pmatrix}2 & 4 & 2\\1 & 5 & 2\\4 & -1 & 9\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\1/2 & 10 & \\2 & -3 & 1\end{pmatrix} \cdot \begin{pmatrix}2 & 4 & 2\\0 & 3 & 1\\0 & 0 & 8\end{pmatrix} = L \cdot U.$$

To solve the equation $Ax = b$, we set $y = Ux$ and use backsolving on $Ly = b$ to see that

$$y = \begin{pmatrix}1\\1/2\\3/2\end{pmatrix}$$

and then again on $Ux = y$ to see that

$$x = \begin{pmatrix}5/48\\5/48\\3/16\end{pmatrix}.$$

2. Problem: Homogeneous and inhomogeneous equations

Consider the system of equations

$$\begin{cases} x_1 & -2x_2 & +x_3 & +x_4 & +2x_5 & = & 2 \\ -x_1 & +3x_2 & & +2x_4 & -2x_5 & = & 2 \\ & +x_2 & +x_3 & +3x_4 & +4x_5 & = & 4 \\ x_1 & +2x_2 & +5x_3 & +13x_4 & +5x_5 & = & 18 \end{cases}$$

- Write the matrix version of this equation as $Ax = b$.
- Write the associated homogeneous equation in matrix form.
- Compute the null space of A.
- Show that b lies in the column space of A. To what matrix equation does this correspond? What particular solution of the original system do you obtain?
- Write the vector form of the general solution to the original system.

2.1. solution.

The matrix version of this equation is

$$\begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 18 \end{pmatrix}.$$

Which has homogeneous form

$$\begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The null space of A is the set of solutions to the homogeneous equation above. Using row reduction, I found that this was equivalent to the system with matrix

$$\begin{pmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Which means that x_3, x_4 are free variables, $x_5 = 0$ and $x_1 = -3x_3 - 7x_4$, $x_2 = -x_3 - 3x_4$. Thus,

$$\text{null}(A) = \left\{ s \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$$

One way to see that b is in the column space of A is to note that b the sum of the third and fourth columns of A. This corresponds to the matrix equation

$$\begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 18 \end{pmatrix},$$

and to the fact that $\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \end{pmatrix}^t$ is a particular solution to $Ax = b$.

The general solution to the inhomogeneous problem $Ax = b$ is

$$S = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$$

3. Problem: On linear dependence

Are the following sets of vectors linearly dependent or linearly independent?

(1) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$

(2) $\begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 18 \end{pmatrix}.$

(3) $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$

3.1. **solution.** To do this: form the matrix with the given columns and compute its null space. If it contains any non-zero vectors, then the set is linearly dependent. If the null space only contains 0, then the set is linearly independent. We find that all three sets are linearly independent.

4. Problem: Determinants

Evaluate the following determinants.

(1) By passing to row echelon form:

$$\begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{vmatrix}$$

(2) By expanding along a row or column and using cofactors:

$$\begin{vmatrix} 3 & 3 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

(3) By whatever method (or combination) seems appropriate:

$$\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}$$

4.1. **solution.** We compute that

$$\begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 4/3 \\ 1 & 0 & 1/3 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

I expanded the second one along its second row:

$$\begin{vmatrix} 3 & 3 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -0\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} = 8 - 6 = 2.$$

For the last one, I used row operations to simplify things and then expanded along the last row and then the first column as shown below:

$$\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$$= -0 + 0 - 0 + 5\begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 5 \cdot 1 \cdot \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = 5 \cdot 4 = 20$$