HOMEWORK ASSIGNMENT # 4

MATH 211, FALL 2006, WILLIAMS COLLEGE

Abstract. These are the instructor's solutions.

1. Problem: Cofactors and Cramer's rule

(1) Use the classical adjoint method to compute the inverse of the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{pmatrix}.$$

(2) Use Cramer's rule to solve the following system.

$$\begin{cases} x_1 & -2x_2 & +x_3 & +x_4 & = & 12 \\ -x_1 & +3x_2 & +x_3 & +2x_4 & = & 12 \\ & +x_2 & +x_3 & +3x_4 & = & 0 \\ x_1 & +2x_2 & +5x_3 & +x_4 & = & 96 \end{cases}$$

1.1. **solution.** Computing cofactors, we see that the classical adjoint of A is

$$\operatorname{adj}(A) = \begin{pmatrix} 11 & -13 & -5 \\ -10 & -10 & 10 \\ -24 & 12 & 0 \end{pmatrix}.$$

So that

$$A^{-1} = \begin{pmatrix} -11/60 & 13/60 & 1/12 \\ 1/6 & 1/6 & -1/6 \\ 2/5 & -1/5 & 0 \end{pmatrix}.$$

To solve the system, we apply Cramer's rule. The matrix form of the system is

$$\begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 0 \\ 96 \end{pmatrix}.$$

The solution is $x = \begin{pmatrix} -11 \\ -3 \\ 24 \\ -7 \end{pmatrix}$.

2. On elementary matrices

Show that any invertible square matrix can be written as a product of elementary matrices.

2.1. Solution. This is a consequence of the Gaussian elimination algorithm. The forward pass corresponds to writing an equation like

$$E_n E_{n-1} \cdots E_2 E_1 A = U$$

where each E_i is an elementary matrix. If A is nonsingular, then U is upper triangular, with nonzero entries down the diagonal. So, further multiplying by elementary matrices converts the pivots to 1's, and eliminates the entries above the pivots, which is must be everything else above the diagonal. We get an equation like

$$F_k F_{k-1} \cdots F_1 E_n \cdots E_1 A = I.$$

So, we invert all of the stuff on the left (working from left to right) to get that

$$A = E_1^{-1} \cdots E_n^{-1} F_1^{-1} \cdots F_{k-1}^{-1} F_k^{-1}.$$

Thus, we have realized A as the product of inverses of elementary matrices. But the inverse of an elementary matrix is still an elementary matrix, because it just realizes the inverse row operation, which is still a row operation. The hard one to see this way is the elimination operation, but you can check this one with a direct computation.

3. On singular matrices

Consider the system $Ax = b$ for

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

Show that the matrix A is singular. One one set of axes, draw a picture of the column space of A in $\mathbb{R}^2$. On another set of axes, draw a picture of the null space of A in $\mathbb{R}^2$ and the solution set to $Ax = b$. Reasoning from the pictures, find a vector b' so that the system $Ax = b'$ has no solution. (Which set of axes should b' live in?)

Can you describe what is happening from the viewpoint of intersecting hyperplanes?

3.1. solution. I'll describe the pictures as best I can. The column space is the line $y = x$. The null space is the line $x = 0$, and the solution set to $Ax = b$ is the line $x = 4$. One vector which does not lie in the column space is $b' = \begin{pmatrix} 1 & 2 \end{pmatrix}^t$, so this vector suffices to answer the next question. Notice that b' should be on the same set of axes as the column space.

From the point of view of intersecting hyperplanes: for a vector $b = \begin{pmatrix} b_1 & b_2 \end{pmatrix}^t$, the solution set to $Ax = b$ is the intersection of the hyperplanes $\mathcal{H}_1 = \{x = b_1\}$ and $\mathcal{H}_2 = \{x = b_2\}$. These are parallel lines in $\mathbb{R}^2$. When they coincide ($b_1 = b_2$), we get a whole line's worth of solutions. When they are different ($b_1 \neq b_2$), we see that there is no solution.