HOMEWORK ASSIGNMENT # 5

MATH 211, FALL 2006, WILLIAMS COLLEGE

Abstract. These are the instructors solutions.

1. Vector Spaces and Subspaces

(1) Show directly that the set V of real valued continuous functions on the interval $[0,1]$ is a vector space under the pointwise operations we discussed in class.

(2) Decide if the following two subsets of V are subspaces. If a subspace, prove it. If not a subspace, say why explicitly.
   - W^{even} is the set of real polynomials of even degree.
   - $W^{\text{even power}}$ is the set of real polynomials in which every term has even degree.

1.1. **Solution.** Let f, g and h be elements of V, and let α and β be real numbers. By the basic algebraic properties of real numbers we have that the following hold for all $x \in [0,1]$:

- $(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$, hence $f+g = g+f$ and vector addition is commutative,
- $(f+(g+h))(x) = f(x)+(g+h)(x) = f(x)+g(x)+h(x) = (f+g)(x)+h(x) = ((f+g)+h)(x)$, hence $f+(g+h) = (f+g)+h$ and vector addition is associative,
- For the function $Z(x) = 0$, we have $(f+Z)(x) = f(x)+0 = f(x)$, so the vector Z is the origin,
- Given a function f, the function $-f$ defined by $(-f)(x) = -f(x)$ satisfies $(f+(-f))(x) = f(x)-f(x) = 0 = Z(x)$, so additive inverses exist,
- $(\alpha(f+g))(x) = \alpha(f+g)(x) = \alpha(f(x)+g(x)) = (\alpha f)(x)+(\alpha g)(x)$, hence scalar multiplication distributes across vector addition,
- $((\alpha+\beta)f)(x) = (\alpha+\beta)f(x) = \alpha f(x)+\beta f(x) = (\alpha f)(x)+(\beta f)(x) = ((\alpha f)+(\beta f))(x)$, hence $(\alpha+\beta)f = (\alpha f)+(\beta f)$ and scalar multiplication distributes across scalar multiplication,
- $((\alpha\beta)f)(x) = (\alpha\beta)f(x) = \alpha(\beta f(x)) = \alpha((\beta f)(x)) = (\alpha(\beta f))(x)$, so $(\alpha beta)f = \alpha(\beta f)$, and, finally,
- $(1f)(x) = 1f(x) = f(x)$, so $1f = f$ and $1 \in \mathbb{R}$ acts as the identity through scalar multiplication.

Therefore, V satisfies the 8 properties of being a vector space. Note that it basically inherits these properties from the algebra of $\mathbb{R}$.

Date: October 23, 2006.

The set W^{even} is not a subspace, as it is not closed under linear combinations. For example, $p = t^2 + t + 1$ and $q = -t^2 + t + 1$ are polynomials of even degree, but their sum $p + q = 2t + 2$ is of odd degree.

The set $W^{\text{even power}}$ is a subspace. Clearly the zero polynomial $Z = 0$ is of degree zero, which is even, and hence the origin is an element of our subset. Next we need to show that the set is closed under linear combination. Two generic elements of $W^{\text{even power}}$ can be written as

$$p = a_{2n}t^{2n} + a_{2n-2}t^{2n-2} + \cdots + a_2t^2 + a_0 \quad \text{and} \quad q = b_{2n}t^{2n} + b_{2n-2}t^{2n-2} + \cdots + b_2t^2 + b_0.$$

So for two scalars α and β, we see that their linear combination

$$\alpha p + \beta q = (\alpha a_{2n} + \beta b_{2n})t^{2n} + (\alpha a_{2n-2} + \beta b_{2n-2})t^{2n-2} + \cdots + (\alpha a_2 + \beta b_2)t^2 + (\alpha a_0 + \beta b_0)$$

also lies in $W^{\text{even power}}$. Hence, $W^{\text{even power}}$ is a subspace.

2. Spans

(1) Find the span of the subset $S = \{w_1, w_2, w_3, w_4\}$ in $\mathbb{R}^3$, where

$$w_1 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \qquad w_3 = \begin{pmatrix} 10 \\ 10 \\ 25 \end{pmatrix}, \qquad w_4 = \begin{pmatrix} 5 \\ -1 \\ 8 \end{pmatrix}.$$

Can you describe this set in a simple, compact way?

(2) Show that the set $S = \{v_1, v_2, v_3\}$ is a linearly independent subset of $\mathbb{R}^4$, where

$$v_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 5 \\ 10 \\ 3 \\ 25 \end{pmatrix}.$$

2.1. **Solution.** We use the row space algorithm to investigate these questions. For the first part, we form the matrix of rows

$$A = \begin{pmatrix} 3 & 1 & 4 \\ 1 & -1 & -1 \\ 10 & 10 & 2 \\ 5 & -1 & 8 \end{pmatrix}.$$

The reduced row echelon form of this matrix is

$$A \sim B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so the vectors must span out all of $\mathbb{R}^3$.

For the second part, we again use the row space algorithm. The matrix of rows is

$$A = \begin{pmatrix} 2 & 3 & 1 & 4 \\ 1 & 1 & -1 & -1 \\ 5 & 10 & 3 & 25 \end{pmatrix}$$

and its reduced row echelon form is the matrix

$$A \sim B = \begin{pmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since each row is non-zero, the rows must be linearly independent, and hence the original vectors must be linearly independent. (Else, they wouldn't span a three dimensional row space.)

3. Bases and dimension

Find the dimension of the intersection of the following collection of hyperplanes in $\mathbb{R}^4$. (Note that they all pass through the origin, so they are subspaces and so is their intersection.) Write down a basis for this intersection.

$$\mathcal{H}_1 = \{w + 2x - y + 3z = 0\}$$
$$\mathcal{H}_2 = \{w - 2x + 10y - 4z = 0\}$$
$$\mathcal{H}_3 = \{5w + 2x + 17y + z = 0\}$$

3.1. **Solution.** This is equivalent to finding a good description for the solution set to a system of homogeneous linear equations. And thus, equivalent to describing the null space of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 1 & -2 & 10 & -4 \\ 5 & 2 & 17 & 1 \end{pmatrix}.$$

The reduced row echelon form of this matrix is

$$A \sim B = \begin{pmatrix} 1 & 0 & 9/2 & -1/2 \\ 0 & 1 & -11/4 & 7/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So we deduce that

$$\text{null}(A) = \text{span}\{\begin{pmatrix} -9/2 & 11/4 & 1 & 0 \end{pmatrix}^t, \begin{pmatrix} 1/2 & -7/4 & 0 & 1 \end{pmatrix}^t\}.$$

As the two vectors involved are linearly independent (clearly–look at the last two coordinates), we see that this is a basis for the span and that the subspace we are interested in has dimension 2.

4. Bases and Coordinates

Consider the vector space $W_3 = \{\text{real polynomials of degree less than or equal to } 3\}$. Show that the set $B = \{1, t, \frac{1}{2}(3t^2 - 1), \frac{1}{2}(5t^3 - 3t)\}$ is a basis of W_3. Write the coordinates of the vector $v = 1 + t + t^2 + t^3$ with respect to this ordered basis.

4.1. **Solution.** First we must show that B is a basis. We check that B is linearly independent. Suppose that

$$a(1) + b(t) + c(\frac{1}{2}(3t^2 - 1)) + d(\frac{1}{2}(5t^3 - 3t)) = 0.$$

Simplifying, we find that

$$(5d/2)t^3 + (c/2)t^2 + (-3d/2 + b)t + (-c/2 + a) = 0.$$

This is equivalent to the system of linear equations

$$\begin{cases} 5d/2 & = & 0 \\ c/2 & = & 0 \\ b - 3d/2 & = & 0 \\ a - c/2 & = & 0 \end{cases}.$$

This clearly has only one solution: $a = b = c = d = 0$. Thus, the set B is linearly independent.

Next, we must check that B spans all of W_3. Let our arbitrary degree three polynomial in t be given by $p = at^3 + bt^2 + ct + d$. We are to find a linear combination of the vectors in B which equals p. It is not too difficult to compute that

$$p = at^3 + bt^2 + ct + d = \frac{2a}{5}(\frac{1}{2}(5t^3 - 3t)) + \frac{2b}{3}(\frac{1}{2}(3t^2 - 1)) + (c + \frac{3a}{5})(t) + (d + \frac{b}{3})(1).$$

Finally, using this representation above with $a = b = c = d = 1$, we can compute that

$$1 + t + t^2 + t^3 = \frac{4}{3}(1) + \frac{8}{5}(t) + \frac{2}{3}(\frac{1}{2}(3t^2 - 1)) + \frac{2}{5}(\frac{1}{2}(5t^3 - 3t)),$$

so that the coordinate representation of our vector is

$$[1 + t + t^2 + t^3]_B = \begin{pmatrix} 4/3 \\ 8/5 \\ 2/3 \\ 2/5 \end{pmatrix}.$$