HOMEWORK ASSIGNMENT # 6 SOLUTIONS

MATH 211, FALL 2006, WILLIAMS COLLEGE

Abstract. These are the instructor's solutions.

1. Rank

Find the rank of the following matrices

$$A = \begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}.$$

1.1. **Solution.** The matrices are row equivalent to the following reduced row echelon forms:

$$A \sim \begin{pmatrix} 1 & 0 & 0 & 22 & -21 \\ 0 & 1 & 0 & -5 & 7 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so they both have rank 3.

2. Reverse engineering

Find a homogeneous system of linear equations whose solution set is spanned by the vectors

$$u_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 4 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 5 \end{pmatrix}.$$

2.1. **Solution.** The following is my argument. For a different solution, see pages 154-155 of your text. We must find equations to describe a subspace which is spanned by u_1, u_2, and u_3. We apply the row space algorithm to find a nicer spanning set (hoping it will be easier to work with). We see

$$\begin{pmatrix} 1 & -2 & 0 & 3 \\ 1 & -1 & -1 & 4 \\ 1 & 0 & -2 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

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Thus, our subspace is

$$\mathcal{S} = \left\{ s \begin{pmatrix} 1 \\ 0 \\ -2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} s \\ t \\ -2s-t \\ 5s+t \end{pmatrix} \right\}.$$

Recalling that the parameters s and t should come from free variables, we see that x_1 and x_2 are free variables, and our system must satisfy equations $x_3 = -2x_1 - x_2$ and $x_4 = 5x_1 + x_2$. That is, our system is

$$\begin{cases} 2x_1 + x_2 + x_3 = 0 \\ 5x_1 + x_2 - x_4 = 0 \end{cases}$$

3. Rows and Columns

Find a basis for the row space and a basis for the column space for each of these matrices.

$$A = \begin{pmatrix} 0 & 0 & 3 & 1 & 4 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 3 \\ 3 & 6 & 5 & 2 & 7 \\ 2 & 4 & 1 & -1 & 0 \end{pmatrix}$$

3.1. **Solution.** Fortunately, both computing a basis for the row space and a basis for the column space require putting the matrix into reduced row echelon form. For A we see that

$$A \sim \begin{pmatrix} 1 & 3 & 0 & 0 & -13/4 \\ 0 & 0 & 1 & 0 & 3/4 \\ 0 & 0 & 0 & 1 & 7/4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

So we deduce that

$$\text{row}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \\ -13/4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3/4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 7/4 \end{pmatrix} \right\},$$

and that

$$\text{col}(A) = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 5 \\ 8 \end{pmatrix} \right\}.$$

Similarly, the reduced row echelon form of B is

$$B \sim \begin{pmatrix} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So we deduce that

$$\text{row}(B) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\},$$

and that

$$\text{col}(B) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 5 \\ 1 \end{pmatrix} \right\}.$$

4. General functions

Suppose that $f : A \to B$ and $g : B \to C$ are functions and that $g \circ f : A \to C$ is surjective. Is it necessary that f is surjective? Is it necessary that g is surjective? If either function must be surjective, give a proof that this is true. If not, give an example of a pair of functions f and g for which the relevant function is not surjective, but the composition still is.

4.1. **Solution.**

(1) First, we show that $g \circ f$ is surjective. Let x be an element of C. We must produce an element a of A such that $g \circ f(a) = c$.

Since g is surjective, there exists a point $b \in B$ such that $g(b) = c$. Since f is surjective, there exists a point $a \in A$ such that $f(a) = b$.

But now $g \circ f(a) = g(f(a)) = g(b) = c$. So we are done.

(2) If $g \circ f$ is surjective, then so must be g, but not necessarily f. To see that f need not be surjective, consider the example of

$$f : [0, 1] \to [0, 1], g : [0, 1] \to [0, 1]$$

defined by $f(x) = x/2$ and

$$g(x) = \begin{cases} 2x, & 0 \le x \le 1/2, \\ 1, & 1/2 < x \le 1. \end{cases}$$

We now prove that g must be surjective (by the *contrapositive*). Suppose that g is not surjective. Then there is a point $c \in C$ such that $g(B)$ does not contain c. But then $(g \circ f)(A) \subseteq g(B)$ does not contain c, and $g \circ f$ is not surjective. This contradicts our hypothesis, so we deduce that g must also be surjective.

5. Linear functions

Let V be the vector space of all smooth (i.e. infinitely many times differentiable) functions $\mathbb{R} \to \mathbb{R}$, W be the vector space of all polynomials with real coefficients in the variable t, and W_3 the vector space of all polynomials of degree at most three in the variable t.

Show that the following mappings are linear maps, and find their kernels and ranges.

(1) $T : W \to W$ defined by $T(p) = t^2 \cdot p$.
(2) $S : W_3 \to W_3$ defined by $S(p) = t \cdot p'$, where p' is the derivative of p with respect to t,
(3) $T_{3,\pi} : V \to W_3$ defined by setting $T_3(f)$ equal to the third degree Taylor polynomial of f centered at the point $a = \pi$.

5.1. **Solution.** It is straightforward to check that these mappings preserve taking linear combinations. The subspaces asked for are as follows: For T,

$$\ker(T) = \{0\}, \quad \operatorname{im}(T) = \operatorname{span}\{t^2, t^3, t^4, \ldots\}.$$

For S,

$$\ker(T) = \operatorname{span}\{1\}, \quad \operatorname{im}(T) = \operatorname{span}\{t, t^2, t^3, t^4, \ldots\}.$$

For $T_{3,\pi}$

$$\ker(T) = \{\text{functions } f \text{ such that} f(\pi) = f'(\pi) = f''(\pi) = f'''(\pi) = 0\}, \quad \operatorname{im}(T) = W_3.$$